Module Application of First Order ODE - Logistic growth model

1. Logistic growth model.

1) Exponential growth.

$$
\begin{aligned}
&y = population \\
&\frac{dy}{dt} = ry \\
&r : rate\, of\, growth\, (r > 0) \\
&\int \frac{1}{y} dy = \int r dt \\
&\ln|y| = rt + C \\
&y = \pm e^C e^{rt} = Ae^{rt} \\
&y(t) = y_0 e^{rt}
\end{aligned}
$$

“Growth rate depend on the population.”

$$r \rightarrow h(y)$$

$$\frac{dy}{dt} = h(y)y$$

Choice of h

① $h(y) \simeq r > 0 \; y \ll 1$

② h decrease as y growth larger.

③ $h > 0, \;\; y \gg 1$

$$h(y) = r - ay \; (a : positive\ constant)$$

$$\frac{dy}{dt} = (r - ay)y)$$

Logistic growth model. (Verhulst equation)

$$\frac{dy}{dt} = r(1 - \frac{q}{r} y)y$$

$$\frac{r}{a} = k$$

$$\frac{dy}{dt} = r(1 - \frac{y}{k})y$$

$r : intrinsic\, growth\, rate$

Simple solution. : $y_0 = const$

$$0 = r(1 - \frac{y}{k})y \quad \text{Equilibrium solution.}$$

$$\Rightarrow \; y = 0 \,\text{or}\, y = k$$

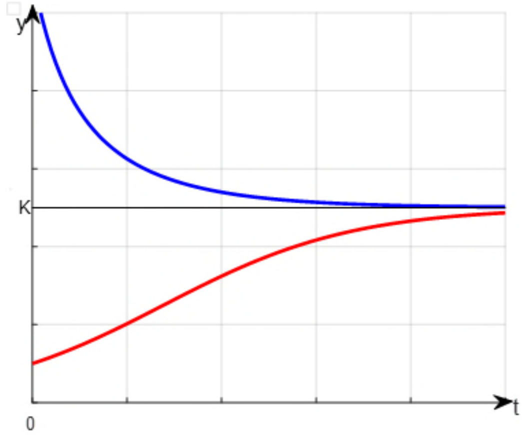

그림 1

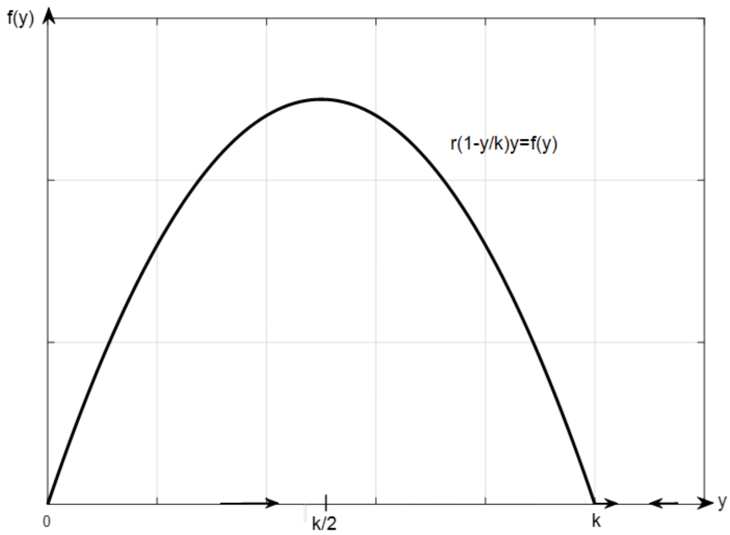

그림 2

$$\frac{dy}{dt} > 0\,,\quad 0 < y < k \quad \text{y moves from left to right}$$

$$\frac{dy}{dt} < 0,\quad y > K \quad \text{y moves from right to left}$$

$$\text{Concavity} \quad \frac{d^2y}{dt^2} = \frac{d}{dt} f(y) = f'(y)\frac{dy}{dt}$$

$$f(y) = r(1 - \frac{y}{k}y) = f'(y)f(y)$$

$$\frac{d^2y}{dt^2} > 0 \quad f' > 0\,\text{and}\,f > 0 \;\Rightarrow\; 0 < y < \frac{k}{2}$$

$$or\,f' > 0\,\text{and}\,f < 0 \;\Rightarrow\; y > k$$

$$\frac{d^2y}{dt^2} < 0 \quad f' > 0 \,\text{and}\, f < 0$$

$$\text{or} f' < 0 \,\text{and}\, f > 0 \quad \frac{k}{2} < y < k$$

k : *Saturation level / Carrying capacity*

$$y_0 < k \;\Rightarrow\; y(t) < k$$

$$\lim_{t\to\infty} y(t) = k$$

How to solve it

$$\int \frac{1}{(1-\frac{y}{k})y} dy = \int r\,dt$$

$$\int \frac{\frac{1}{k}}{1-\frac{y}{k}} + \frac{1}{y} = ry + C$$

$$\ln|y| - \ln\left|1-\frac{y}{k}\right| = rt + C$$

$$\ln\left|\frac{y}{1-\frac{y}{k}}\right| = rt + C$$

$$\frac{y}{1-\frac{y}{k}} = \pm e^c e = Ae^{rt}$$

$$\frac{y_0}{1-\frac{y_0}{k}} = A$$

$$\Rightarrow \frac{y}{1-\frac{y}{k}} = \frac{y_0}{1-\frac{y_0}{k}} e^{rt}$$

$$y = X(1-\frac{y}{k})$$

$$(1+\frac{1}{k}X)y = X$$

$$y = \frac{X}{1+\frac{1}{k}X} = \frac{1}{\frac{1}{k}+\frac{1}{X}}$$

$$y = \frac{1}{\frac{1}{k} + \frac{1-\frac{y_0}{k}}{y_0} e^{-rt}}$$

$$= \frac{y_0 k}{y_0 + (k-y_0)e^{-rt}}$$

$$\lim_{t\to\infty} y(t) = \frac{y_0 k}{y_0} = k$$

Q: Are $y=0$ and $y=k$ singular solutions?

2. Modified logistic model

Fox squirrel is a small mammal native to the Rocky Mt. They are very territorial. So, if their population is large, their rate of growth decreases and become negative. On the other hand. If the population is too small, fertile adults run the risk of not being able to find suitable mates, so again the rate of growth is negative.

⇒ Model

N: *Carrying capacity* → Too big population.

M: *Sparsity consistant* → Too small population.

$$\frac{dy}{dt} = g(y)$$

$$\begin{aligned} g'(y) < 0 &\quad \text{if } \ y < N \\ &\quad \ \ or \ \ y < M \\ g'(y) > 0 &\quad \text{if } \ M < y < N \end{aligned}$$

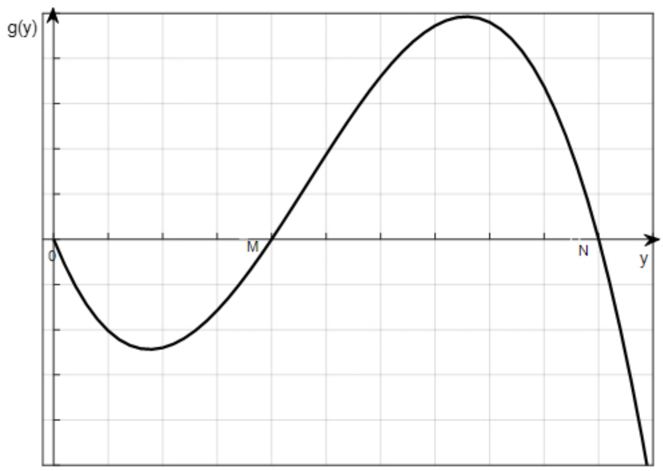

그림 3

$$\frac{dy}{dt} = g(y) = ky(1-\frac{y}{N})(somthing)$$

$$\begin{aligned} (something) > 0 &\quad y > M \\ (something) < 0 &\quad y < M \end{aligned}$$

$$(somthing) = (\frac{y}{M} - 1)$$

$$\frac{dy}{dt} = ky(1-\frac{y}{N})(\frac{y}{M}-1) \quad (k > 0)$$

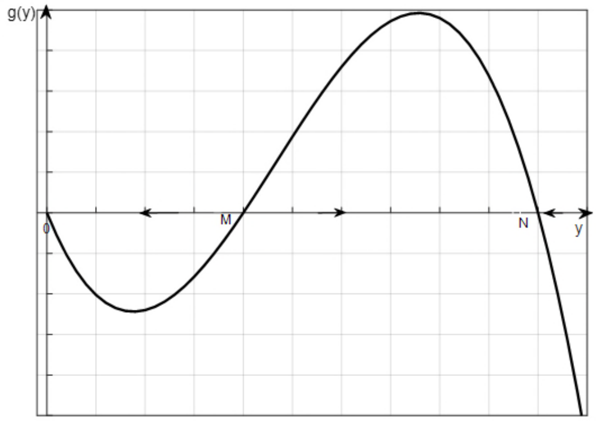

그림 4

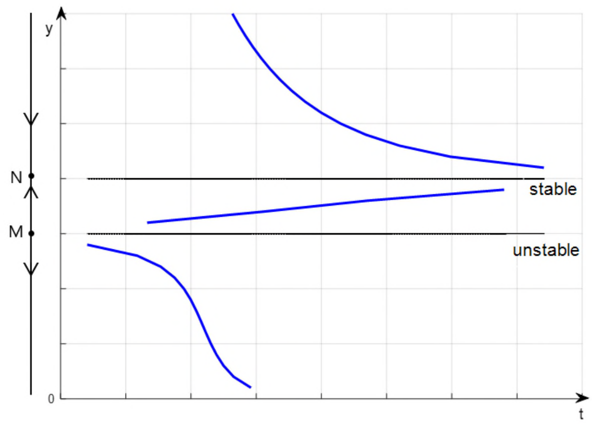

그림 5