Module System of linear ODEs - Phase diagram을 이용한 해의 분석

1. System of LDE

Motivation: for geometric strategy of linear system.

1) Two CD stores.

$x(t) =$ Daily profit of store A at time t.

$y(t) =$ Daily profit of store B at time t.

$x(t) > 0$ Making money.

$x(t) < 0$ Losing money.

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dx}{dt} = cx + dy \end{cases}$$

$$x > 0 \quad a > 0$$
$$y > 0 \quad b > 0$$

$y > 0 \quad b < 0 \quad$ Store B steals customers from A.

2) Two CD store problem.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}$$

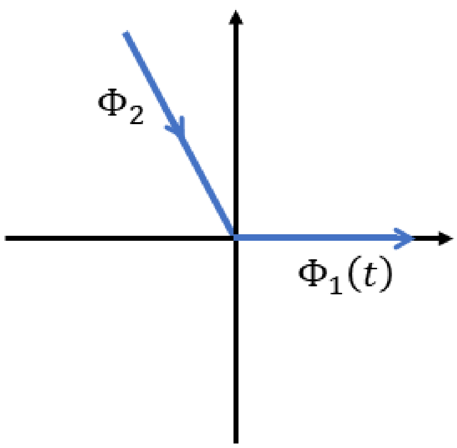

그림 1

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix} \quad \frac{d}{dt}(4\Phi_2) = A4\Phi_2$$

$$\text{If}\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{Try}\ \Phi_1 + \Phi_2 &= \begin{bmatrix} e^{2t} - e^{-4t} \\ 2e^{-4t} \end{bmatrix} \\ &= \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

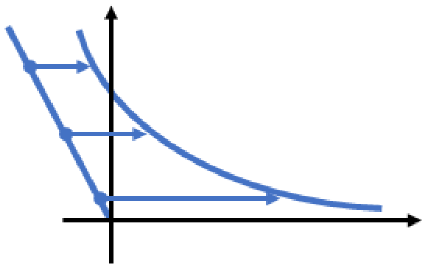

그림 2

3) Two burger shops problem.

$$A = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix}$$

$$x_1{}' = 2x_1 + ax_2$$
$$x_2{}' = x_2$$

$$\lambda = 2, \mu = 1$$

$$\lambda = 2, \quad \begin{bmatrix} 0 & a \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mu = 1, \quad \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ -1 \end{bmatrix}$$

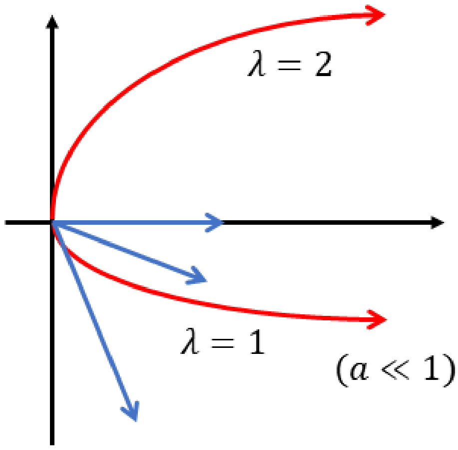

그림 3

$$A = \begin{bmatrix} 2 & a \\ b & 1 \end{bmatrix}$$

$$(2-\lambda)^2(1-\lambda) - ab = 0$$
$$\lambda^2 - 3\lambda + 2 - ab = 0$$
$$\lambda = \frac{1}{2}(2 \pm \sqrt{1+4ab})$$
$$a = 1, \;\; 1 + 4b > 9, \;\; b > 2$$

$$1 + 4ab = 16, \quad 4ab = 15, \quad ab = \frac{15}{4}$$

$$a = \frac{5}{4}, b = 3$$

$$\begin{bmatrix} 2 & \frac{5}{4} \\ 3 & 1 \end{bmatrix}$$

$$\lambda = \frac{1}{2}(3 \pm 4) = \frac{7}{2}, -\frac{1}{2}$$

$$\lambda = \frac{7}{2}, \quad \begin{bmatrix} 2 - \frac{7}{2} & \frac{5}{4} \\ 3 & 1 - \frac{7}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{4} \\ 3 & -\frac{5}{2} \end{bmatrix} \sim \begin{bmatrix} -6 & 5 \\ -6 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\mu = -\frac{1}{2}, \quad \begin{bmatrix} 2+\frac{1}{2} & \frac{5}{4} \\ 3 & 1+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{5}{4} \\ 3 & \frac{3}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

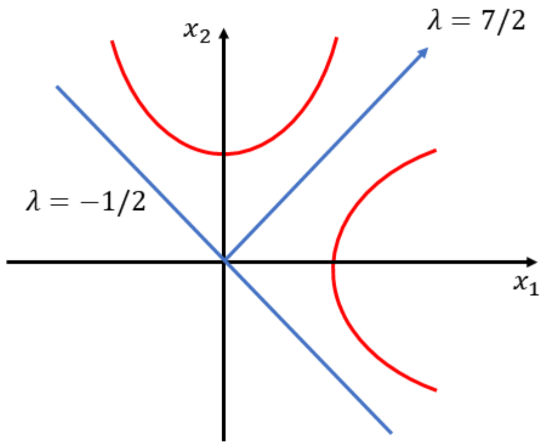

그림 4

4) Phase plane/ Phase portrait of linear system.

$\lambda < \mu < 0$	sink(nodal)
$\lambda > \mu > 0$	source(nodal)
$\lambda < 0 < \mu$	saddle
$\lambda = \mu$	improper/()? node

$\lambda, \overline{\lambda}$:complex

$Re(\lambda) > 0$	simple source
$Re(\lambda) < 0$	spiral source
$Re(\lambda) = 0$	center

Ex) Nodal sink.

$$A = \begin{bmatrix} -6 & -2 \\ 5 & 1 \end{bmatrix} \qquad \begin{array}{l} \lambda = -4, -1 \\ \overrightarrow{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \end{array}$$

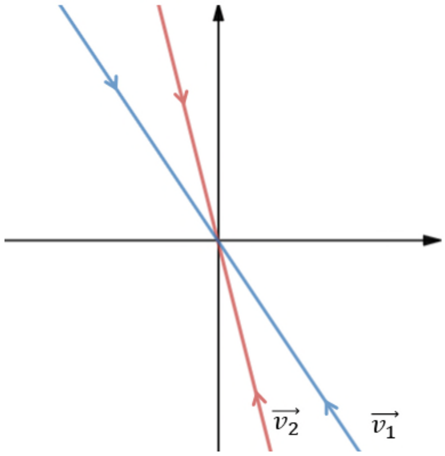

그림 5

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{d}{dt}\vec{x}(0) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$$

$$\text{at } \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} -3-2 \\ \frac{5}{2}+1 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{7}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad 5a+b=0 \quad b=-5a$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} -6-2a \\ 5a+b \end{bmatrix} \qquad 5a+b<0 \quad b=5a$$

$$\begin{aligned} \vec{x}(t) &= x_1 \boldsymbol{v_1} e^{-4t} + c_2 \boldsymbol{v_2} e^{-t} \\ &= e^{-t}\left(c_1 \boldsymbol{v_1} e^{-3t} + c_2 \boldsymbol{v_2}\right) \end{aligned}$$

$$\to c_2 \boldsymbol{v_2} e^{-t}$$

Source

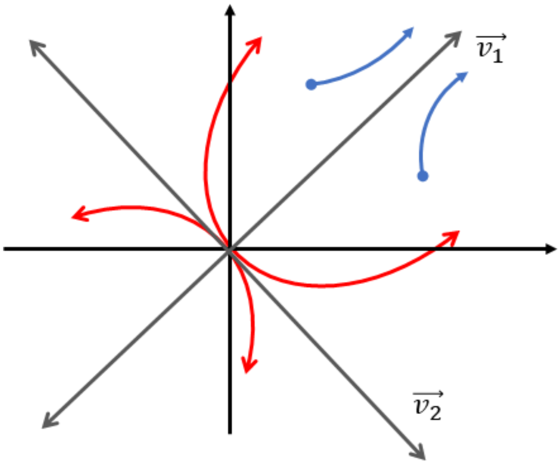

그림 6

$\lambda_1 > \lambda_2 > 0$

$\vec{x}(t) = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_2}e^{\lambda_2 t}$

$= e^{\lambda_1 t}\left[c_1\vec{v_1} + c_2\vec{v_2}e^{(\lambda_2 - \lambda_1)t}\right]$

$-(\lambda_1 - \lambda_2) < 0$

$\to c_1\vec{v_1}e^{\lambda_1 t}$

Saddle

$$A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix} \qquad \lambda = 1, -5$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

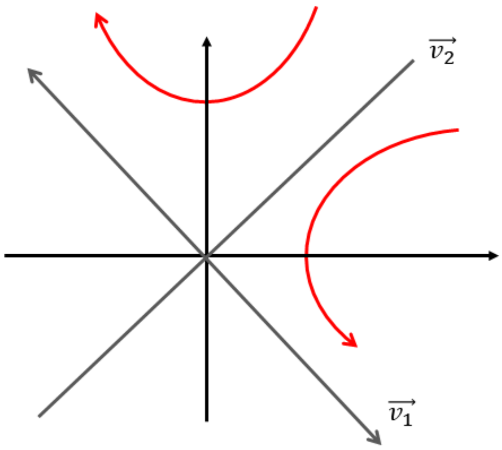

그림 7

Two → Store

Overcrowding effect.

$x_1(0) = x_2(0)$

$x_1(0) > x_2(0)$

$x_1(0) < x_2(0)$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} -2a-3b \\ -3a-2b \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -1 \end{bmatrix} \to \begin{bmatrix} -2+3 \\ -3+2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$2a+3b=0 \qquad b=-\frac{2}{3}a$$

2. improper node

Ex)

$$A = \begin{bmatrix} -10 & 6 \\ -6 & 2 \end{bmatrix} \qquad \lambda = -4 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A + 41)w = v$$

$$w = \begin{bmatrix} 1 \\ \frac{7}{6} \end{bmatrix}$$

$$\overrightarrow{x}(t) = c_1 \overrightarrow{v_1} e^{-4t} + c_2 (t\overrightarrow{v} + \overrightarrow{w}) e^{-4t}$$

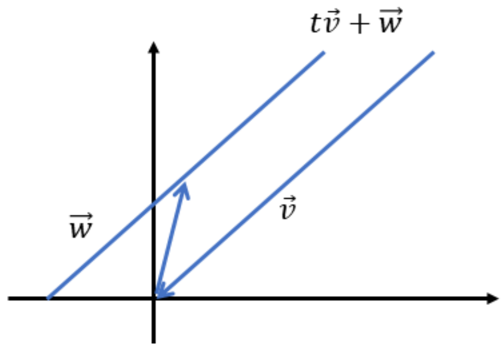

그림 8

$(\overrightarrow{v}, \overrightarrow{w}) > 0$ → Clockwise

$(\overrightarrow{v}, \overrightarrow{w}) < 0$ → Counter clockwise

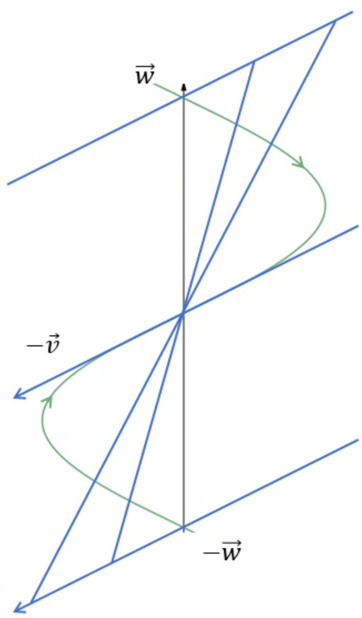

그림 9

$$(t\overrightarrow{v} + \overrightarrow{w}) e^{-4t}$$

Curve transverses line.

$$\left[\left(c_1 \overrightarrow{v_1} + c_2 \overrightarrow{w} \right) + c_2 \overrightarrow{v} t \right] e^{-4t}$$

Ex)

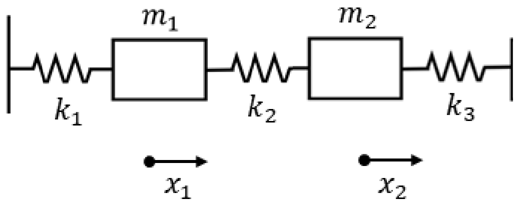

그림 10

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{bmatrix} \qquad \boldsymbol{X}' = \begin{bmatrix} x_1' \\ x_1'' \\ x_2' \\ x_2'' \end{bmatrix} = \begin{bmatrix} x_1' \\ -\frac{(k_1+k_2)}{m_1}x_1 + \frac{k_2}{m_1}x_2 \\ x_2' \\ \frac{k_2}{m_2}x_1 - \frac{(k_2+k_3)}{m_2}x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{(k_2+k_3)}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{bmatrix}$$

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ a & -\lambda & b & 0 \\ 0 & 0 & -\lambda & 1 \\ c & 0 & d & -\lambda \end{bmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & b & 0 \\ 0 & -\lambda & 1 \\ 0 & d & -\lambda \end{vmatrix} - \begin{vmatrix} a & b & 0 \\ 0 & -\lambda & 1 \\ c & d & -\lambda \end{vmatrix} \\ &= (-\lambda)^2 \begin{vmatrix} -\lambda & 1 \\ d & -\lambda \end{vmatrix} - \left\{ a \begin{vmatrix} -\lambda & 1 \\ d & -\lambda \end{vmatrix} \right\} - b \begin{vmatrix} 0 & 1 \\ c & -\lambda \end{vmatrix} \\ &= (-\lambda)^2(\lambda^2 - d) - \{a\lambda^2 - ad + bc\} \\ &= \lambda^4 - d\lambda^2 - a\lambda^2 + ad - bc \\ &= \lambda^4 - (a+d)\lambda^2 + ad - bc = 0 \end{aligned}$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda^2 I \right) = 0$$

Case : $m_1 = m_2, \quad k_1 = k_2 = k_3$

$$a = -\frac{2k}{m}, b = \frac{k}{m}, c = \frac{k}{m}, d = -\frac{2k}{m}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\lambda^4 + \frac{4k}{m}\lambda^2 + (4-1)\left(\frac{k}{m}\right)^2 = 0$$
$$\left(\lambda^2 + \frac{3k}{m}\right)\left(\lambda^2 + \frac{k}{m}\right) = 0$$

$$\lambda = \pm i\sqrt{3}\,\omega_0$$
$$\pm i\omega_0$$
$$\omega_0 = \sqrt{\frac{k}{m}}$$
$$\xi = U + iV$$
$$\xi e^{i\omega_0 t}, \bar{\xi} e^{-i\omega_0 t}$$
$$\eta e^{i\sqrt{3}\,\omega_0 t}, \eta e^{-i\sqrt{3}\,\omega_0 t}$$

$$\xi = \xi_1 + i\xi_2$$
$$\Phi_1 = \xi_1 \cos\omega_0 t - \xi_2 \sin\omega_0 t$$
$$\Phi_2 = \xi_1 \sin\omega_0 t + i_2 \cos\omega_0 t$$
$$\Phi_3 = \eta_1 \cos\sqrt{3}\,\omega_0 t - \eta_2 \sin\sqrt{3}\,\omega_0 t$$
$$\Phi_4 = \eta_1 \sin\sqrt{(3)\omega_0}\,t + \eta_2 \cos\sqrt{(3)\omega_0}\,t$$

$$c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 + c_4\Phi_4$$

$$\xi = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1/\omega_0 \\ 0 \\ 1/\omega_0 \\ 0 \end{bmatrix}$$

$$\Phi_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cos\omega_0 t - \begin{bmatrix} 1/\omega_0 \\ 0 \\ 1/\omega_0 \\ 0 \end{bmatrix} \sin\omega_0 t$$

$$\begin{bmatrix} x_1 \\ x_{1'} \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega_0} sin\omega_0 t \\ -\cos\omega_0 t \end{bmatrix} \qquad x_1 + x_2 = 0$$

$$\begin{bmatrix} x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -\frac{1}{\omega_0} sin\omega_0 t \\ \cos\omega_0 t \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ -2\omega_0^2 & -\lambda & \omega_0^2 & 0 \\ 0 & 0 & -\lambda & 1 \\ \omega_0^2 & 0 & -2\omega_0^2 & -\lambda \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} -i\omega_0 & 1 & 0 & 0 \\ -2\omega_0^2 & -i\omega_0 & \omega_0^2 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ \omega_0^2 & 0 & -2\omega_0^2 & -i\omega_0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2\omega_0^2 & 2i\,\omega_0 & 0 & 0 \\ -2\omega_0^2 & -i\omega_0 & \omega_0^2 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ 0 & -i\,\omega_0 & -2\omega_0^2 & -i\omega_0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -i\omega_0 & 1 & 0 & 0 \\ 0 & i & \omega_0 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ 0 & -i & -2\omega_0 & -i \end{bmatrix}$$

$$\sim \begin{bmatrix} \omega_0 & i & 0 & 0 \\ -2\omega_0 & -i & \omega_0 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ \omega_0 & 0 & -2\omega_0 & -i \end{bmatrix}$$

$$\sim \begin{bmatrix} \omega_0 & i & 0 & 0 \\ 0 & i & \omega_0 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ 0 & 0 & -\omega_0 & -i \end{bmatrix}$$

$$i\omega_0 v_3 = v_4$$
$$iv_2 = -\omega_0 v_3$$
$$\omega_0 v_1 = -iv_2$$
$$v_3 = -\frac{t}{i\omega_0}$$
$$v_2 = -\frac{\omega_0}{i}\left(\frac{-t}{i\omega_0}\right)$$
$$v_1 = -\frac{i}{\omega_0}(-t)$$
$$= \frac{\quad}{\omega_0}$$

$$\sim \begin{bmatrix} \omega_0 & i & 0 & 0 \\ 0 & i & \omega_0 & 0 \\ 0 & 0 & -i\omega_0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \sin\omega_0 t + \begin{bmatrix} 1/\omega_0 \\ 0 \\ 1/\omega_0 \\ 0 \end{bmatrix} \cos\omega_0 t$$

$$= \begin{bmatrix} 1/\omega_0 \cos\omega_0 t \\ -\sin\omega_0 t \\ 1/\omega_0 \cos\omega_0 t \\ \sin\omega_0 t \end{bmatrix}$$

$$x_1 = x_2$$

Fundamental models.

$$\lambda = i\sqrt{3}\,\omega_0$$

$$\begin{bmatrix} \sqrt{3}\,\omega_0 & i & 0 & 0 \\ -2\omega_0 & -\sqrt{3}\,i & \omega_0 & 0 \\ 0 & 0 & -i\sqrt{3}\,\omega_0 & 1 \\ \omega_0 & 0 & -2\omega_0 & -i\sqrt{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} \sqrt{3}\,\omega_0 & i & 0 & 0 \\ 0 & \left(\frac{2}{\sqrt{3}} - \sqrt{3}\right)i & \omega_0 & 0 \\ 0 & 0 & -i\sqrt{3} & 1 \\ 0 & -\frac{i}{\sqrt{3}} & -2\omega_0 & -i\sqrt{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} \sqrt{3\omega_0} & i & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}i & \omega_0 & 0 \\ 0 & 0 & -i\sqrt{3}\,\omega_0 & 1 \\ 0 & 0 & -3\omega_0 & -i\omega_0 \end{bmatrix}$$

$$\sim \begin{bmatrix} * \\ * \\ 0 \; 0 \; -i\sqrt{3}\,\omega_0 \; 1 \\ 0 \; 0 \; 0 \; 0 \end{bmatrix}$$

$$i\sqrt{3}\,\omega_0 w_3 = w_4 \qquad w_3 = \frac{t}{i\sqrt{3}\,\omega_0}$$
$$w_4 = t$$

$$\frac{1}{\sqrt{3}} i w_3 = \omega_0 w_3$$

$$w_2 = \frac{\sqrt{3}\,\omega_0}{i} \frac{t}{i\sqrt{3}\,\omega_0} = -t$$

$$w_1 = \frac{-i}{\sqrt{3}\,\omega_0}(-t) = \frac{it}{\sqrt{3}\,\omega_0}$$

$$\overrightarrow{w} = \begin{bmatrix} \frac{i}{\sqrt{3}\,\omega_0} \\ -1 \\ \frac{1}{i\sqrt{3}\,\omega_0} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} \frac{1}{\sqrt{3}\,\omega_0} \\ 0 \\ \frac{1}{\sqrt{3}\,\omega_0} \\ 0 \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} x_1 \\ x_1{}' \\ x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 0 \\ -\cos\sqrt{3}\,\omega_0 t \\ 0 \\ \cos\sqrt{3}\,\omega_0 t \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{3}\,\omega_0}\sin\sqrt{3}\,\omega_0 t \\ 0 \\ \frac{1}{\sqrt{3}\,\omega_0}\sin\sqrt{3}\,\omega_0 t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}\,\omega_0}\sin\sqrt{3}\,\omega_0 t \\ -\cos\sqrt{3}\,\omega_0 t \\ \frac{1}{\sqrt{3}\,\omega_0}\sin\sqrt{3}\,\omega_0 t \\ \cos\sqrt{3}\,\omega_0 t \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$(\sqrt{3}\,\omega_0)^2 + x_2^2 = 1$$