# Module System of linear ODEs - 복소 고유값과 중복 고유값 경우

## 1) 선형연립미분방정식의 응용

Ex)

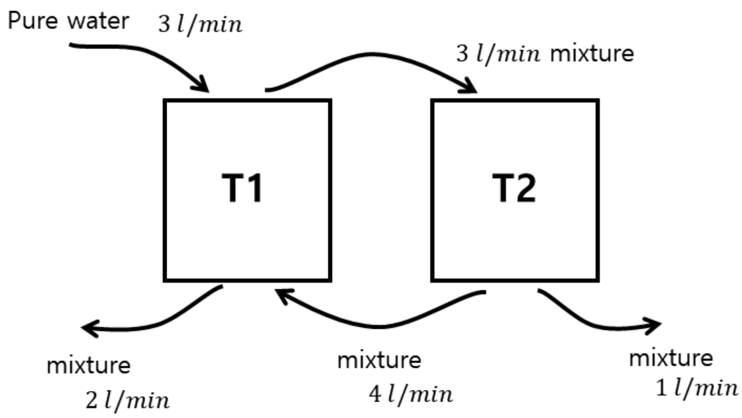


그림 1

$$\text{T1: } Water(t=0)=20\,l$$
$$Chlorine(t=0)=150g$$
$$\text{T2: } W(t=0)=10t$$
$$C(t=0)=50g$$

Determine the amount of chlorine in each tank at any time t>0.

Sol)

$$x_j(t) = amount\,of\,chlorine \in \tan k_j (\in y)$$

$$\varDelta x_1 = rate\ \text{in} - rate\,out$$

$$= (3l/\min)(0y/l)(\triangle t\min) + (3l/\min)(\frac{x_2}{10}g/l)(\triangle t\min)$$

$$-(2l/\min)(\frac{x_2}{10}g/l)(\triangle t\min) - (4l/\min)(\frac{x_2}{20}g/l)(\triangle t\min)$$

$$x_1{}'(t) = \frac{3}{10}x_2 - \frac{6}{20}x_1$$

$$\triangle x_2(t) = \in - out$$

$$\approx\ (4l/\min)(\frac{x_1}{20}g/l)(\triangle t\min)$$

$$-(3l/\min)(\frac{x_2}{10}g/l)(\triangle t\min)$$

$$-(1l/\min)(\frac{x_2}{10}g/l)(\triangle t\min)$$

$$x_2{}'(t) = \frac{1}{5}x_1 - \frac{4}{10}x_2$$

$$x'(0) = Ax = \begin{bmatrix} -\frac{3}{10} & \frac{3}{10} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} x$$

$$x(0) = \begin{bmatrix} 150 \\ -50 \end{bmatrix}$$

$$\begin{vmatrix} -3/10-\lambda & 3/10 \\ 1/5 & -1/3-\lambda \end{vmatrix} = 0$$

$$\left(-\frac{3}{10}-\lambda\right)\left(-\frac{2}{5}-\lambda\right)-\frac{3}{50} = 0$$

$$\lambda = -\frac{1}{10}, -\frac{3}{10}$$

$$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Omega = \begin{bmatrix} 3/2e^{-\frac{1}{10}t} & -e^{-\frac{3}{5}t} \\ e^{\frac{t}{10}} & e^{-\frac{3}{5}t} \end{bmatrix}$$

$$C_1 = \frac{20}{20} = 1$$

$$C_2 = \frac{30}{10} = 3$$

$$\frac{C_1}{C_2} = \frac{Q_1/20}{Q_2/10} = \frac{Q_1}{Q_2} \times \frac{1}{2}$$

2) Complex eigenvalues

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}\vec{x}(t)$$

$$\left(-\frac{1}{2}+\lambda\right)^2 + 1 = 0$$

$$\pm\frac{1}{2}+\lambda = \pm i$$

$$\lambda = -\frac{1}{2} \pm i$$

$$\Phi(t) = \vec{v}e^{\lambda t}$$

$$\lambda = -\frac{1}{2}+i$$

$$\begin{bmatrix} -\frac{1}{2}-(-\frac{1}{2}+i) & 1 \\ -1 & -\frac{1}{2}-(-\frac{1}{2}+i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$-iv_1 + v_2 = 0 \qquad v_2 = iv_1 \qquad \vec{v} = v_1\begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$v_1 + iv_2 = 0$$

$$A\begin{bmatrix} 1 \\ i \end{bmatrix} = \left(-\frac{1}{2}+i\right)\begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$A\begin{bmatrix} 1 \\ -i \end{bmatrix} = \left(-\frac{1}{2}-i\right)\}$$

$$\Phi_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}e^{(-1/2+i)t}$$
$$\Phi_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}e^{(-1/2-i)t}$$

$$\Phi_1 = e^{-1/2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(\cos t + i\sin t)$$
$$= e^{-1/2t}\left[\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + i\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right)\right]$$

$$\Phi_2 = \Phi_1$$

$$\Phi_1 = e^{-1/2t}(\Psi_1 + i\Psi_2)$$
$$\Phi_2 = e^{-1/2t}(\Psi_1 - i\Psi_2)$$

$$e^{-1/2t}\Psi_1 = RE\Phi_1 = \frac{\Phi_1+\Phi_2}{2} = \frac{1}{2}(\Phi_1+\Phi_2)$$
$$e^{-1/2t}\Psi_2 = IM\Phi_1 = \frac{\Phi_1-\Phi_2}{2i} = \frac{1}{2i}(\Phi_1-\Phi_2)$$

Fundamental set: $e^{-1/2t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, e^{-1/2t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

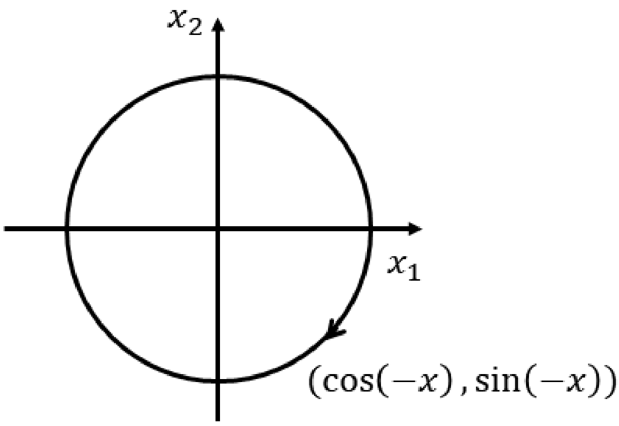


그림 2


그림 3

$$x_1^2 + x_2^2 = e^{-t}$$
$$r(t) = e^{-\frac{1}{2}t}$$

Ex)

$$m, c, k \quad m = 1$$
$$c^2 - 4k < 0$$
$$k = 1, c = 1 \qquad 1 - 4 < 0$$
$$y'' - y' + y = 0$$
$$y_1 = y \qquad {y_1}' = y' = y_2$$
$$y_1 = y' \qquad {y_2}' = -y' - y = -y_2 - y_1$$
$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\vec{y}(t)$$
$$\begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-1-\lambda)+1=0$$

$$\lambda^2+\lambda+1=\left(\lambda+\frac{1}{2}\right)^2+\frac{3}{4}=0$$

$$\lambda+\frac{1}{2}=\pm i\frac{\sqrt{3}}{2}$$

$$\lambda=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\text{Eigenvector}\begin{bmatrix}\frac{1}{2}-i\frac{\sqrt{3}}{2} & 1\\ -1 & -\frac{1}{2}-i\frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0$$

$$\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)v_1+v_2=0$$

$$\begin{bmatrix}1\\ -\frac{1}{2}+i\frac{\sqrt{3}}{2}\end{bmatrix}\text{ or }\begin{bmatrix}2\\ -1+i\sqrt{3}\end{bmatrix}$$

$$\Phi_1=e^{-1/2t}\left(\begin{bmatrix}2\\ -1\end{bmatrix}+i\begin{bmatrix}0\\ \sqrt{3}\end{bmatrix}\right)\left(\cos\frac{\sqrt{3}}{2}t+i\sin\frac{\sqrt{3}}{2}t\right)$$

$$=e^{-1/2t}\left[\left(\begin{bmatrix}2\\ -1\end{bmatrix}\cos\frac{\sqrt{3}}{2}t-\begin{bmatrix}0\\ \sqrt{3}\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)+i\left(\begin{bmatrix}0\\ \sqrt{3}\end{bmatrix}\cos\frac{\sqrt{3}}{2}t+\begin{bmatrix}2\\ -1\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)\right]$$

$$\Psi_1=e^{-1/2t}\begin{bmatrix}2\cos\frac{\sqrt{3}}{2}t\\ -\cos\frac{\sqrt{3}}{2}t-\sqrt{3}\,sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$

$$\Psi_2=e^{-1/2t}\begin{bmatrix}2\sin\frac{\sqrt{3}}{2}t\\ \sqrt{3}\,cos\frac{\sqrt{3}}{2}t-\sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$

$$\begin{bmatrix}y\\ y'\end{bmatrix}=c_1\Psi_1+c_2\Psi_2$$

$$y=e^{-1/2t}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\sin\frac{\sqrt{3}}{2}t\right)$$

$$\begin{bmatrix}y\\ y'\end{bmatrix}=\begin{bmatrix}2\cos\frac{\sqrt{3}}{2}t\\ -\cos\frac{\sqrt{3}}{2}t-3\sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$

$$\left(\frac{y}{2}\right)^2+\left(\frac{y'+y/2}{\sqrt{3}}\right)^2=1$$

$$\begin{bmatrix}2 & 0\\ -1 & -\sqrt{3}\end{bmatrix}\begin{bmatrix}\cos\frac{\sqrt{3}}{2}t\\ \sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$

$$\begin{bmatrix}1\\ 0\end{bmatrix}\rightarrow\begin{bmatrix}2\\ -1\end{bmatrix}\quad\begin{bmatrix}0\\ 1\end{bmatrix}\rightarrow\begin{bmatrix}0\\ -\sqrt{3}\end{bmatrix}$$

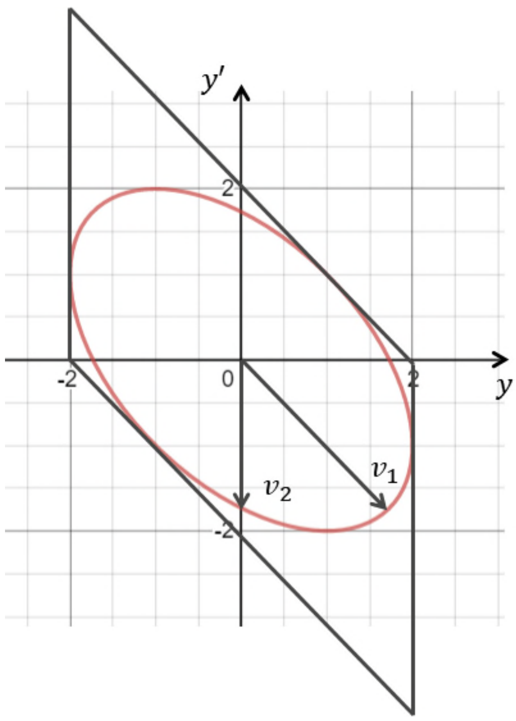


그림 **4**

**3) Repeated Eigenvalues/ multiplicity**

Case :

$X' = AX$ does not have n linearly independent eigenvectors.

$$X' = AX \qquad A = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$$

$$|A - \lambda I_2| = \begin{bmatrix} 1-\lambda & 3 \\ -3 & 7-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(7-\lambda) + 9 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

$$\begin{bmatrix} 1-4 & 3 \\ -3 & 3 \end{bmatrix} \qquad \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Phi_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \qquad E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Try $\Phi_2(t) = X_1 te^{3t} + E_2 e^{4t}$ $\qquad$ ($E_2 e^{4t}$: correction term)

$$\Phi_2{}' = E_1 e^{4t} + 4E_1 te^{4t} + 4E_2 e^{4t}$$

$$\Phi_2{}' - A(E_1 te^{4t} + E_2 e^{4t}) = (4E_1 - AE_1)te^{4t} + (E_1 + 4E_2 - AE_2)e^{4t}$$

$$E_1 + 4E_2 - AE_2 = 0$$
$$(A - 4I)E_2 = E_1$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-3a + 3b = 1$$

$$a = s,\ b = \frac{3s+1}{3}, \qquad \begin{bmatrix} s \\ \frac{3s+1}{3} \end{bmatrix}$$

$$s = 1 \Rightarrow \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix} e^{4t} = \begin{bmatrix} 1+t \\ \frac{4}{3}+t \end{bmatrix} e^{4t}$$

Ex)

$$A = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} (-2-\lambda) & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

$$(\lambda+2)^3 = \lambda^3 + 6\lambda^2 + 3 \cdot \ 4\lambda + 8$$

Eigenvector $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} = E_1$

$$\Phi_1 = E_1 e^{-2t}$$
$$\Phi_2 = E_1 te^{-2t} + E_2 e^{-2t}$$
$$\Phi_2{}' - A\Phi_2 = E_1 e^{-2t} - 2E_1 te^{-2t} - 2E_2 e^{-2t} - \left(AE_1 te^{-2t} + AE_2 e^{-2t}\right)$$

$$\begin{vmatrix} -2-\lambda & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & -3-\lambda \end{vmatrix} = -(2+\lambda)\begin{vmatrix} -(7+\lambda) & 0 \\ 1 & 3-\lambda \end{vmatrix} - 25\begin{vmatrix} -1 & -5 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2+\lambda)(7+\lambda)(3-\lambda) - 25(\lambda-3+5)$$

$$= (\lambda+2)\{(21-25+(3-7)\lambda-\lambda^2)\}$$

$$= -(\lambda+2)(\lambda+2)^2$$

$$= -(\lambda+2)^3 \qquad \lambda = -2$$

$$\begin{bmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad \begin{matrix} y+5z=0 \\ 5x=y \end{matrix} \qquad \begin{bmatrix} \frac{1}{5}y \\ y \\ -\frac{1}{5}y \end{bmatrix}$$

$$(A+2)E_2 = E_1$$

$$E_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \qquad \begin{matrix} y+5z=-1 \\ 5x-y=1 \end{matrix} \qquad y=0 \qquad E_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix}$$

$$(A+2)E_3 = E_2 \quad y = 5z = -\frac{1}{5}$$

$$25x-5y=0 \quad x=y=0 \quad z=-\frac{1}{25}$$

$$E_3 = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{25} \end{bmatrix}$$

$$\varPhi_1 = e^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

$$\varPhi_2 = te^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix}$$

$$\varPhi_3 = \frac{1}{2}t^2e^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} + te^{-2t}\begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{25} \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} \frac{1}{2}t^2+\frac{1}{5}t \\ \frac{5}{2}t^2 \\ -\frac{1}{2}t^2-\frac{1}{25} \end{bmatrix}$$

$$= te^{-2t}(-2E_1-AE_1) + (E_1-2E_1-AE_2)e^{-2t}$$

$$(A+2I)E_2 = E_1$$

$$E_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

$$\Phi_2 = E_1 te^{-2t} + E_2 e^{-2t} = \begin{bmatrix} -1-t \\ -4-5t \\ 1+t \end{bmatrix} e^{-2t}$$

$$\Phi_3 = \frac{1}{2} E_1 t^2 e^{-2t} + E_2 te^{-2t} + E_3 e^{-2t}$$

$$\Phi_3{}' = \left[ E_1 re^{-2t} - E_2 t^2 e^{-2t} + E_2 e^{-2t} - 2E_2 te^{-2t} - 3E_3 3^{-2t} \right]$$

$$\Phi_3{}' = A\Phi_3 = A\left(\frac{1}{2}E_1 t^2 + E_2 t + E_3\right)e^{-2t}$$

$$\begin{cases} -E_1 = \frac{1}{2}AE_1 \\ E_1 - 2E_2 = AE_2 \end{cases}$$

$$E_2 - 2E_3 = AE_3$$

$$\Rightarrow (A+2)E_3 = E_2 \qquad E_3 = \begin{bmatrix} -\frac{24}{25} \\ -4 \\ 1 \end{bmatrix}$$

$$\Phi_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -\frac{24}{25} \\ -4 \\ 1 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -\frac{24}{25} - t - \frac{1}{2}t^2 \\ -4 - 4t - \frac{3}{2}t^2 \\ 1 + t + \frac{1}{2}t^2 \end{bmatrix} e^{-2t}$$

Question) How can we diagonalize A?

$$AP = PD$$

$$P = [v_1 \cdots \ v_n]$$

$$Av_j = P\begin{bmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{bmatrix} = \lambda_j v_j$$

$$\Rightarrow D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \lambda_j : eigenvalues\, of\, A$$

$P = [v_1^{-1} \cdots \ v_n^{-1}]$ $\qquad$ $v_j^{-1}$: Eigenvectors associated $\lambda_j$

Ex)

$$\boldsymbol{x}' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}\boldsymbol{x}$$

$$\begin{vmatrix} 3-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - 3 = 0$$

$$\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2) = 0$$
$$\lambda = 2, 6$$
$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ex)

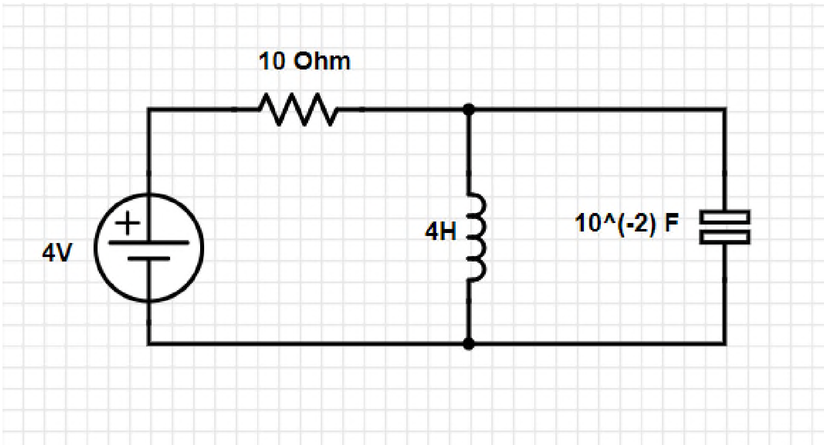


**그림 5**

Two internal loop and one external loop.

$$\begin{cases} 10i_1 + 4(i_1{}' - i_2{}') = 4 \\ 10i_1 + 100q_2 = 4 \end{cases}$$
$$\Rightarrow \begin{cases} i_1{}' = -10i_2 \\ 2(i_1{}' - i_2{}') = -5i_1 + 2 \end{cases}$$
$$\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} i_1{}' \\ i_2{}' \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} i_1{}' \\ i_2{}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -10 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\boldsymbol{i} = A\boldsymbol{i} + G$$
$$A = \begin{bmatrix} 0 & -10 \\ \frac{5}{2} & -10 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$|A - \lambda I| = -\lambda(-10 - \lambda) + 25 = 0$$
$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0$$
$$\Phi_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}$$
$$\Phi_2 = E_1 t e^{-5t} + E_2 e^{-5t}$$
$$\Phi_2{}' = A\Phi_2 \quad \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 5/2 & -5 \end{bmatrix} E_2$$
$$\Phi_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 3/10 \end{pmatrix} e^{-5t}$$

$$\Omega(t) = \begin{bmatrix} 2e^{-5t} & (1+2t)e^{-5t} \\ e^{-5t} & (\frac{3}{10}+t)e^{-5t} \end{bmatrix}$$

$$\Omega^{-1} = e^{5t} \begin{bmatrix} -\frac{1}{4}(3+10t) & \frac{3}{2}(1+2t) \\ \frac{5}{2} & -5 \end{bmatrix}$$

$$U(t) = \int \Omega^{-1}(t) G(t) dt$$

$$= \begin{bmatrix} \int -\frac{5}{2}(1+2t)e^{5t} dt \\ \int 5e^{5t} dt \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{10}e^{3t} - e^{3t}t \\ e^{5t} \end{bmatrix}$$

$$\Psi_1 = \Omega(t) U(t) = \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix}$$

$$\boldsymbol{i}(t) = \Omega C + \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$$

$$\boldsymbol{i}(0) = \begin{pmatrix} 2/5 \\ 2/5 \end{pmatrix}$$

Initial condition $i_1 - i_2 = 0$ on $L$.

$$10 i_1(0) + 100 q_2(0) = 4 \quad i_1(0) = \frac{2}{5}$$

$$-\begin{bmatrix} 2/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 2/5 \end{bmatrix} = \Omega(0) C \qquad C = \Omega^{-1} \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{i}(t) = \Omega \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$$