Module System of linear ODEs - 동차 경우의 일반해

1. Review on matrix (Linear algebra)

$A=(a_{ij})$ n×m matrix

i th row $[a_{i1} \cdots \ a_{im}]$

ith column $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$

$$A^T=(a_{ji})$$

$$A^*=A^{-T}=(\overline{a_{ji}})$$

$$A=\begin{bmatrix} 3 & 1+2i \\ 2-i & -4 \end{bmatrix} \qquad A^*=\begin{bmatrix} \overline{3} & \overline{2-i} \\ \overline{1-2i} & \overline{-4} \end{bmatrix}$$
$$=\begin{bmatrix} 3 & 2+i \\ 1-3i & -4 \end{bmatrix}$$

Addition

$$A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$$
$$A+B=B+A$$
$$A+(B+C)=(A+B)+C$$

Multiply by scalar

$$\alpha A=\alpha(a_{ij})=(\alpha a_{ij})$$

Multiplication

$$AB=\underset{n\times m}{(a_{ij})}\underset{m\times p}{(b_{ij})}=(\sum a_{ik}b_{kj})$$

$I_n=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ Identity matrix

$$AI_n=I_nA=A \qquad A_{n\times m}$$

$$AB=BA=I$$

$$B=A^{-1} \text{ Inverse of } A$$

$$\Leftrightarrow \det \neq 0$$

$$A\vec{x}=\vec{b} \quad A: n\times n$$

$$a_{11}x_1+\cdots \ a_{1n}x_n=b_1$$
$$\vdots$$
$$a_{n1}x_1+\cdots \ a_{nn}x_n=b_n$$

$$A=(a_{ij}) \quad \vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix} \quad \vec{b}=\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

A Is invertible $\Rightarrow$ $\vec{x}=A\vec{b}$

$A\vec{x}=\vec{0}$(homogeneous equation) has only trivial solution $\vec{x}=0$

1) Linearly independence.

$$v, w \quad w \neq v$$

v_1, v_2, v_3 no redundant vector

$$v_2 \neq cv_1 \text{ and } v_3 \neq \alpha_1 w_1 + \beta v_2$$

(characterize) $v_1, \cdots, v_n$ linearly independent

$$\Leftrightarrow c_1 v_1 + \cdots + c_n v_n = 0$$
$$\Rightarrow c_1 = \cdots = c_n = 0$$

Ex)

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} -4 \\ 1 \\ -11 \end{bmatrix}$$

Determine whether they are linearly independent or not.

Search for non-zero c_1, c_2, c_3 .

$$\begin{bmatrix} 1 & 2 & 04 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = ①$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 + 2c_2 - 4c_3 = 0$$
$$c_2 - 3c_3 = 0$$

$$c_3 = t \quad c_2 = 3t$$
$$c_1 = -2c_2 + 4c_3$$
$$= -6t + 4t = -2t$$

$$\begin{bmatrix} -2t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow -2v_1 + 3v_2 + v_3 = 0$$

$$\det[v_1 \ v_2 \ v_3] = 0$$

$$\det[v_1 \ v_2 \ \alpha_1 v_1 + \alpha_2 v_2]$$
$$= \alpha_1 \det[v_1 \ v_2 \ v_3] + \alpha_2 \det[v_1 \ v_2 \ v_3]$$

행렬식 계산법 ① $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = a_1 \begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v_1, \cdots, v_n \in R \qquad \text{linearly independent}$$

$$w \in R^n \Rightarrow \exists\ \alpha_{1,} \cdots, \alpha_2 \in R$$

$$\text{such that } w = \sum_{j=1}^{n} \alpha_j v_j$$

따라서 방정식 $A = \left[\overrightarrow{v_1} \cdots \overrightarrow{v_n}\right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = w$ 은 항상 유일한 해를 가짐. 역행렬을 이용하여 해를 표현하면

$$\vec{\alpha} = A^{-1}\vec{w} \quad \Rightarrow \quad \det(A) \neq 0$$

$\vec{v}, \cdots, \overrightarrow{v_n}$ lineally independent $\Rightarrow \det(v_1 \cdots v_n) \neq 0$

$$\det[v_1, \cdots, v_n] = \det \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= \det \begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = d_{n,} \cdots, d_n \neq 0$$

2. 선형 연립 미분방정식 개관

Homogeneous Linear system with constant coefficients

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$x_1 = c_1 e^{2t}$$

$$x_2 = c_2 e^{-3t}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

$$\underset{\Phi_1}{\underset{\parallel}{e^{2t}\overrightarrow{v_1}}} \quad \underset{\Phi_2}{\underset{\parallel}{e^{-3t}\overrightarrow{v_2}}}$$

Ex)

$$x_1{}'(t) = 3x_1 + 3x_2 + 8$$

$$x_2{}'(t) = x_1 + 5x_2 + 4e^{3t}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

In general $X' = AX + G,\ X(t_0) = X^0$

Theorem.

$I \ni t_0$
suppose $a_{ij}(t), g_j(t)$ are continuous on I.
then Initial Value Problem.
$\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{G},\ \boldsymbol{X}(t_0) = \boldsymbol{X}^0$ has a unique solution defined at all $t \in I$

Homogeneous system

$$\boldsymbol{X}' = A\boldsymbol{X}$$

Ex)

$$\boldsymbol{A}' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} \boldsymbol{X}$$

$\boldsymbol{\Phi}_1(t) = \begin{pmatrix} -2e^{35} \\ e^{3t} \end{pmatrix}$ $\boldsymbol{\Phi}_2(t) = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$ defined on $\boldsymbol{R}$

Linearly independent.

$$\boldsymbol{\Phi}_3(t) = \begin{pmatrix} (11-6t)e^{35} \\ (-4+3t)e^{3t} \end{pmatrix} = -4\boldsymbol{\Phi}_1 + 3\boldsymbol{\Phi}_2$$

Theorem.

$$\boldsymbol{\Phi}_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \vdots \\ \psi_{n1} \end{pmatrix}, \cdots, \boldsymbol{\Phi}_n = \begin{pmatrix} \psi_{1n} \\ \vdots \\ \psi_{nn} \end{pmatrix}$$

are solution of $\boldsymbol{X}' = A\boldsymbol{X}$ on $\boldsymbol{I}$

$\boldsymbol{\Phi}_1, \cdots, \boldsymbol{\Phi}_n$ are linearly independent on $\boldsymbol{I}$ if and only if $\boldsymbol{\Phi}_1(t_0), \cdots, \boldsymbol{\Phi}_n(t_0)$ are linearly independent.

that is $\begin{vmatrix} \psi_{11}(t_0) & \psi_{12}(t_0) & \cdots & \psi_{1n}(t_0) \\ \psi_{21}(t_0) & \psi_{22}(t_0) & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \psi_{n1}(t_0) & \psi_{n2}(t_0) & \cdots & \psi_{nn}(t_0) \end{vmatrix} \neq 0$.

Ex)

$$\boldsymbol{\Phi}_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix},\ \boldsymbol{\Phi}_2(t) = \begin{pmatrix} (1-13t)e^{3t} \\ te^{3t} \end{pmatrix}$$

$$\boldsymbol{\Phi}_1(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ \boldsymbol{\Phi}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = -1-1 \neq 0$$

Nonlinear pt: $\Rightarrow$

Suppose $\boldsymbol{\Phi}_1(t_0), \cdots, \boldsymbol{\Phi}_n(t_0)$ are linearly independent. want to show that $\boldsymbol{\Phi}_1(t), \cdots, \boldsymbol{\Phi}_n(t)$ are linearly independent.

Suppose $\boldsymbol{\Phi}_1(t_0) = c_2\boldsymbol{\Phi}_2(t_0) + \cdots + c_n\boldsymbol{\Phi}_n(t_0)$.

Define $\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}_1 - c_2\boldsymbol{\Phi}_2 - \cdots - c_n\boldsymbol{\Phi}_n \Rightarrow \boldsymbol{\Phi}$ is solution of $\boldsymbol{X}' = A\boldsymbol{X}$.

$$\boldsymbol{\Phi}(t_0) = 0 .$$

$\boldsymbol{\Phi}$ is solution of IVP $\boldsymbol{X}' = A\boldsymbol{X}, \boldsymbol{X}(t_0) = 0$

But $\boldsymbol{\Psi}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is also solution on $\boldsymbol{I}$.

$$\Rightarrow \boldsymbol{\Phi}(t) = 0 \text{ on } \boldsymbol{I}$$

$$\Rightarrow \boldsymbol{\Phi}_1(t) = \sum_{j=2}^{n} c_j \boldsymbol{\Phi}_j \text{ on } \boldsymbol{I}$$

Theorem.

1. $\boldsymbol{X}' = A\boldsymbol{X}$ has n linearly independent solutions defined on $\boldsymbol{I}$.
2. Given n linearly independent solutions $\boldsymbol{\Phi}_1(t), \cdots, \boldsymbol{\Phi}_n(t)$ defined on $\boldsymbol{I}$, every solution on $\boldsymbol{I}$ is a linear combination of $\boldsymbol{\Phi}_1(t), \cdots, \boldsymbol{\Phi}_n(t)$.\

"General solution" $\sum_{j=1}^{n} c_j \boldsymbol{\Phi}_j(t)$

Ex)

$$\boldsymbol{X}' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} \boldsymbol{X}$$

$\boldsymbol{\Phi}_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}, \boldsymbol{\Phi}_2(t) = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$

$\boldsymbol{\Phi} = c_1\boldsymbol{\Phi}_1 + c_2\boldsymbol{\Phi}_2 = \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \boldsymbol{\Omega} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\boldsymbol{\Omega}$ is called "fundamental matrix" of $\boldsymbol{X}' = A\boldsymbol{X}$.

$$\boldsymbol{X}(t) = \boldsymbol{\Omega}(t)\boldsymbol{C}$$

2. Homogeneous case 일반해 구하기.

A is constant matrix.

$$\boldsymbol{X} = \xi e^{\lambda t}$$

$$\lambda \xi = A \xi$$

Find λ and ξ satisfying such λ is called eigenvalue of A and ξ is called eigenvector of λ.

Theorem.

Suppose A has $\lambda_1, \cdots, \lambda_n$ eigenvalues and associated eigenvectors $\xi_1, \cdots, \xi_n$ that linearly independent.

$\Rightarrow$ $\xi_1 e^{\lambda_1 t}, \cdots, \xi_n e^{\lambda_n t}$ are linearly independent solutions of $\boldsymbol{A}' = A\boldsymbol{X}$ on $\boldsymbol{R}$

Ex)

$$\boldsymbol{X}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \boldsymbol{X}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 6 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 1, 6$$

$$A\boldsymbol{X} = \lambda \boldsymbol{X}$$

$$(A - \lambda I)\boldsymbol{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boldsymbol{X}$ is nonzero vector.

$\Rightarrow$ $A - \lambda I$ is not invertible.

$\Rightarrow$ $\det(A - \lambda I) = 0$

Ex)

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 9 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 16 = 0$$

$$1 - \lambda = \pm 4$$

$$\begin{aligned} \lambda &= 1 \pm 4 \\ &= 5, -3 \end{aligned}$$

Eigenvector for $\lambda = 5$.

$$\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \vec{v} = 5\vec{v}$$

$$\begin{bmatrix} 1-5 & 2 \\ 8 & 1-5 \end{bmatrix} \vec{v} = \vec{0}$$

$$\begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix} \vec{v} = \vec{0}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ (또는 } \vec{v}\text{의 상수배)}$$

Eigenvector for $\lambda = -3$.

$$\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \vec{w} = -3\vec{w}$$

$$\begin{bmatrix} 1+3 & 2 \\ 8 & 1+3 \end{bmatrix} \vec{w} = \vec{0}, \quad \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \vec{w} = \vec{0}$$

$$\vec{w} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \text{(또는 } \vec{w}\text{의 상수배)}$$

$$\text{일반해는 } \vec{X}(t) = c_1 \vec{v} e^{5t} + c_2 \vec{w} e^{-3t} = c_1 \begin{bmatrix} 2e^{5t} \\ 4e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-3t} \\ -4e^{-3t} \end{bmatrix}$$

Ex)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} \\
&= \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 0 & -(1+\lambda) & -(1+\lambda) \end{vmatrix} \\
&= -(l+\lambda) \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 0 & 1 & 1 \end{vmatrix} \\
&= -(1+\lambda)\left\{(-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\right\} \\
&= -(1+\lambda)\{(-\lambda)(-\lambda+1) - (1+1)\} \\
&= -(1+\lambda)[\lambda^2 - \lambda - 2] \\
&= -(1+\lambda)(\lambda-2)(\lambda+1)
\end{aligned}$$

$$\lambda = -1, 2$$

$$\lambda = -1$$

$$A\vec{v} = -\vec{v} \quad (A+I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \vec{v} = \vec{0}$$

$$v_1 + v_2 - v_3 = 0$$

Set $v_2 = t, v_3 = s$, then $v_1 = -t + s$.

$$\vec{v} = \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 2$$

$$A\vec{w} = 2\vec{w} \quad (A-2I)\vec{w} = \vec{0}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}\vec{w} = \vec{0}$$

$$\begin{bmatrix} 0 & -3 & -3 \\ 1 & -2 & -1 \\ 0 & -3 & -3 \end{bmatrix}\vec{w} = \vec{0}$$

$$\begin{cases} w_1 - 2w_2 - w_3 = 0 \\ \qquad w_2 + w_3 = 0 \end{cases}$$

$w_3 = t$ 라 두면, $w_2 = -t$

$$x_1 = 2w_2 + w_3$$
$$= -2t + t = -t$$

$$\vec{w} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

take $\vec{w} = \begin{Bmatrix} -1 \\ -1 \\ 1 \end{Bmatrix}$

$$\Phi_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\Phi_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$\Phi_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$