Module Laplace transform - 2계 선형 미분방정식에 응용

1) Solving D.E with discontinuous forcing function.

Ex)

$$2y'' + y' + 2y = g(t)$$

$$g(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0 & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

$$y(0) = y'(0) = 0$$

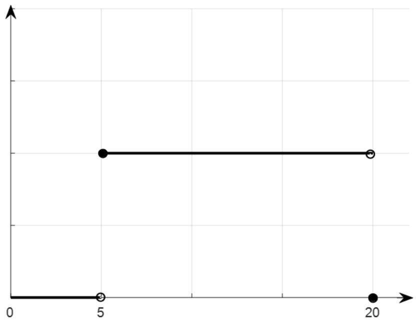

그림 1

$$y = u_5(t) - u_{20}(t)$$

$$\mathcal{L}\ [2y'' + y' + 2y] = \mathcal{L}\ [g]$$

$$2(s^2 Y - sy(0) - y'(0)) + sY - y(0) + 2Y = \frac{1}{s}e^{-5s} - \frac{1}{s}e^{-20s}$$

$$(2s^2 + s + 2)Y = \frac{1}{s}(e^{-5s} - e^{-20s})$$

$$Y = \left[e^{-cs} \frac{1}{s(2s^2+s+2)} \right] \qquad c = 5, 20$$

$$\mathcal{L}\ [u_c(s) f(t+c)] = e^{-cs}\mathcal{L}\ [f](s)$$

$$f = ? \qquad \mathcal{L}^{-1}\left[\frac{1}{s(2s^2+s+2)} \right]$$

$$2s^2 + s + 2 = 2(s^2 + \frac{1}{2}s + 1)$$

$$= 2[(s + \frac{1}{4})^2 + \frac{15}{16}]$$

$$\frac{A}{C} = \frac{Bs + C}{2s^2 + s + 2}$$

$$A(2s^2 + s + 2) + s(Bs + C) = 1$$

$$(2A + B)s^2 + (A + C)s + 2A = 1$$

$$1 + B = 0 \qquad B = -1, \qquad C = -\frac{1}{2}$$

$$f=\mathcal{L}^{-1}\left[-\frac{1/2}{s}\right]+\mathcal{L}^{-1}\left[\frac{-s-1/2}{2s^2+s+2}\right]=\frac{1}{2}-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{s+1/4}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right]-\frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right]$$

$$\frac{1}{2}\frac{s+1/2}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}=\frac{1}{2}\left[\frac{s+1/4}{(s+\frac{1}{4})^2+\frac{15}{16}}+\frac{1/4}{(s+\frac{1}{4})^2+\frac{15}{16}}\right]$$

$$\mathcal{L}\left[e^{a}f\right]=F(s-a)$$

$$\mathcal{L}\left[\cos\frac{\sqrt{15}}{4}t\right]=\frac{s}{s^2+\frac{15}{16}}$$

$$\mathcal{L}\left[\cos\frac{\sqrt{15}}{4}t\right]=\frac{s}{s^2+\frac{15}{16}}$$

$$\mathcal{L}\left[\sin\frac{\sqrt{15}}{4}t\right]=\frac{\sqrt{15}/4}{s^2+15/16}$$

$$f=\frac{1}{2}-\frac{1}{2}e^{-1/4t}\cos\frac{\sqrt{15}}{14}t-\frac{1}{8}\frac{4}{\sqrt{15}}\sin\frac{\sqrt{15}}{4}t$$

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-cs}\frac{1}{s(2s^2+s+2)}\right]&=u_c(t)f(t-c)\\&=u_c(t)\left(\frac{1}{2}-\frac{1}{2}e^{-1/4(t-c)}\cos\left(\frac{\sqrt{15}}{4}(t-c)\right)-\frac{1}{2\sqrt{15}}e^{-1/4(t-c)}\sin\left(\frac{\sqrt{15}}{4}(t-c)\right)\right)\end{aligned}$$

$$y=u_5(t)f(s-5)-u_{20}f(t-20)$$

Ex)

$$y''+4y=g(t)$$

$$y(0)=0\quad y'(0)=0$$

$$y(t)=\begin{cases}0 & 0\le t<5\\(t-5)/5 & 5\le t<10\\1 & t\ge 10\end{cases}$$

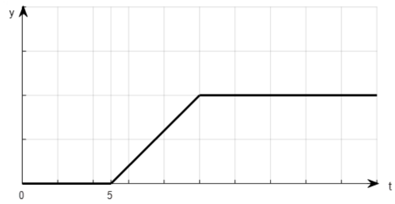

그림 2

$$s^2Y+4Y=\mathcal{L}[g]$$

$$Y=\frac{1}{s^2+4}\mathcal{L}[g]$$

$$\begin{aligned}g&=(u_5(t)-u_{10}(t))\frac{t-5}{5}+u_{10}(t)\\&=u_5(t)\frac{t-5}{5}-u_{10}(t)\left(\frac{t-5}{5}-1\right)\end{aligned}$$

$$Y=\frac{\left(e^{-5s}-e^{-10s}\right)}{5s^2(s^2+4)}$$

$$H=\frac{1}{s^2(s^2+4)}\xrightarrow{\mathcal{L}^{-1}}h$$

$$y=\frac{1}{5}\left(u_5(t)h(t-5)-u_{10}(t)h(t-10)\right)$$

$$H=\frac{1}{4}\left(\frac{1}{s^2}-\frac{1}{s^2+4}\right)$$

$$h=\frac{1}{4}\left(t-\frac{1}{2}\sin 2t\right)$$

$$\begin{aligned} t>10 \quad y&=\frac{1}{5}\left[h(t-5)-h(t-10)\right] \\ &=\frac{1}{2}\left[(t-5)-\frac{1}{2}\sin(2(t-5)-(t-10)+\frac{1}{2}\sin(2(t-10))\right] \\ &=\frac{1}{20}\left[5-\frac{1}{2}\sin(2(t-5))+\frac{1}{2}\sin(2(t-10))\right] \end{aligned}$$

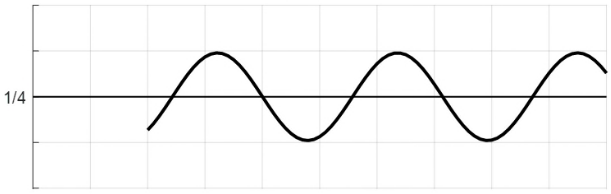

그림 3

$-\sin A+\sin B$

$$=-\left(\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right)+\left(\sin\frac{A+B}{2}\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right)$$

$$=-2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$-\sin 2(t-5)+\sin 2(t-10)=-2\cos(2t-15)\sin 5$$

$$y=\frac{1}{4}-\frac{1}{20}\cos(2t-15)\sin 5$$

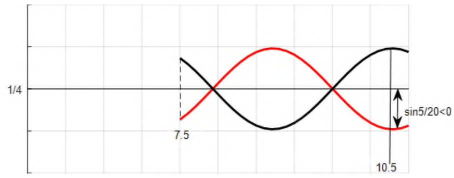

그림 4

2) Impulse function. (충격 함수)

$$ay''+by'+cy=g(t)$$

$$g(t)=\left\{large-\right.$$

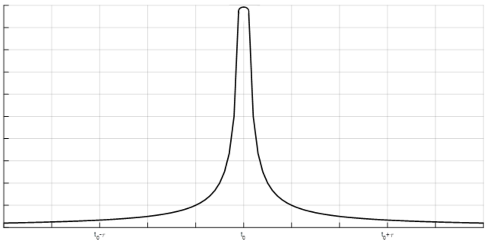

그림 5

$$\mathcal{L}\ [\tau] = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt$$

(Stretching of forcing function.)

"Total impulse" of $g(t)$ on $|t-t_0| < \tau$.

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & -\tau < t < \tau \\ 0 & otherwise \end{cases}$$

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)dt = 1$$

Def)

Unit impulse function δ. (Dirac delta)

$$\delta(t) := \lim_{\tau \to 0+} d_\tau(t)$$

$$\delta(t-t_0) = \lim_{\tau \to 0+} d_\tau(t-t_0)$$

$$\mathcal{L}\ [\delta(t-t_0)] := \lim_{\tau \to 0} \mathcal{L}\ [d_\tau(t-t_0)]$$

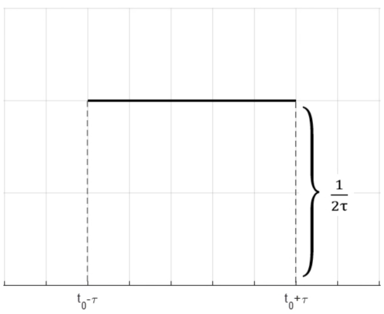

그림 6

$$\Rightarrow d_\tau(t-t_0) = \frac{1}{\tau}(u_{t_0-\tau}(t) - u_{t_0+\tau}(t))$$

$$\mathcal{L}\ [d_\tau(t-t_0)] = \frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s} - \frac{e^{-(t_0+\tau)s}}{s}\right)$$

$$= e^{-t_0 s}\left(\frac{e^{\tau s} - e^{-\tau s}}{2\tau s}\right)$$

$$\begin{aligned}\lim_{\tau\to 0}\mathcal{L}\ [d_\tau(t-t_0)] &= e^{-t_0 s}\lim_{\tau\to 0}\frac{e^{\tau s}-e^{-\tau s}}{2\tau s}\\ &= e^{-t_0 s}\lim_{\tau\to 0}\frac{se^{\tau s}+se^{-\tau s}}{2s}\\ &= e^{-t_0 s}\frac{1}{2}(e^0+e^0)\\ &= e^{-st_0}\end{aligned}$$

Claim $\mathcal{L}\ [\delta(t-t_0)]=e^{-st_0}$.

Ex)

$$y''+2y'+2y=\delta(t-\pi)$$

$$y(0)=y'(0)=0$$

$$s^2Y+2sY=2Y=e^{-\pi s}$$

$$Y=\frac{1}{s^2+2s+2}e^{-\pi s}$$

$$y=u_\pi(t)f(t-\tau)$$

$$f(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right]=e^{-t}\sin t$$

$$\frac{1}{s^2+2s+2}=\frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\ [\sin t]=\frac{1}{s^2+1}$$

$$\mathcal{L}\ [e^{-t}\sin t]=\frac{1}{(s+1)^2+1}$$

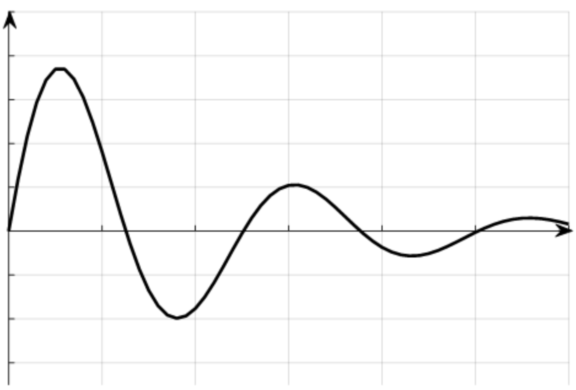

그림 7

$$\text{If } g=\delta(x-1)$$

$$u=\delta_1*\frac{1}{2\sqrt{3\pi t}}e^{-\frac{(x-1)^2}{12t}}$$

$$y(t)=u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$

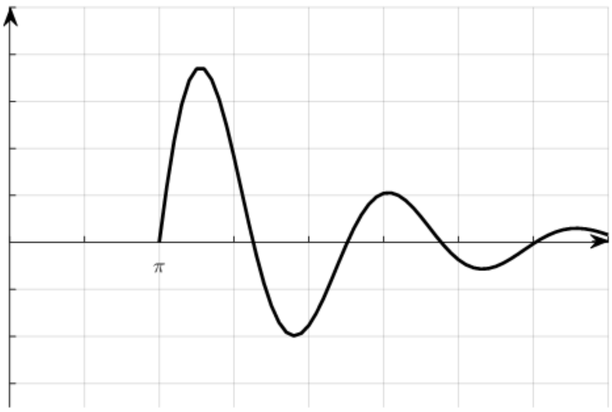

그림 8

3) Convolution

$$y'' + Ay' + By = f(t)$$
$$\to \mathcal{L}\ [y'' + Ay' + Bt] = F(s)$$
$$\to\ Y(s) = \frac{1}{s^2+1}F(s) + \cdots$$

$$y(t) = \mathcal{L}^{\ -1}\left[\frac{1}{s^2+1}F(s)\right] + \mathcal{L}^{\ -1}[\cdots]$$

Def)

f, g Defined on $[0, \infty)$.

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

Theorem.

$$\mathcal{L}\ [f*g] = \mathcal{L}\ [f]\mathcal{L}\ [g]$$

pf) let $F = \mathcal{L}\ [f], G = \mathcal{L}\ [g]$

$$F(s) = G(s) = F(s)\int_0^\infty e^{-st}g(\tau)d\tau$$
$$= \int_0^\infty e^{-st}F(s)g(\tau)d\tau$$

$$\mathcal{L}\ [H(t-\tau)f(t-\tau)] = e^{-\tau s}F(s)$$

$$F(s)G(s) = \int_0^\infty \mathcal{L}\ [H(t-\tau)f(t-\tau)](s)g(\tau)d\tau$$
$$= \int_0^\infty \left[\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)dt\right]g(\tau)d\tau$$
$$= \int_0^\infty \int_0^\infty e^{-st}g(\tau)H(t-\tau)f(t-\tau)dtd\tau$$
$$= \int_0^\infty \int_\tau^\infty e^{-st}g(\tau)f(t-\tau)dtd\tau$$

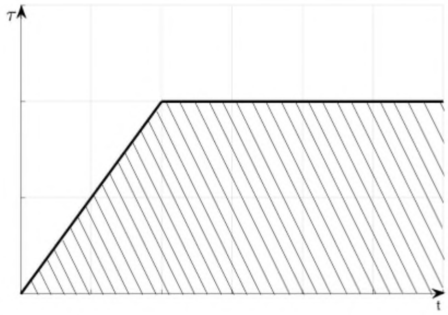

그림 9

$$\int_0^\infty \left[\int_0^t g(\tau)f(t-\tau)d\tau\right]e^{-st}dt = \int_0^\infty e^{-st}f*g(t)$$

Ex)

$$\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right] = te^{4t}$$

$$\downarrow \quad F = \frac{1}{s},\ G = \frac{1}{(s-4)^2}$$

$$f = 1 \quad ,\ g = te^{4t}$$

ANS)

$$\begin{aligned} f*g &= \int_0^t \tau e^{4\tau}d\tau \\ &= \frac{1}{4}\tau e^{4\tau}|_0^t - \int_0^t \frac{1}{4}e^{4\tau}d\tau \\ &= \frac{1}{4}te^{4t} - \frac{1}{16}(e^{4t}-1) \end{aligned}$$

Theorem.

$$f*g = g*f$$

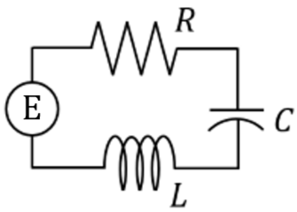

그림 10

$$L\frac{di}{dt}+Rt(t)+\frac{q(t)}{C}=E(t)$$

$$q(t)=\int_0^t\frac{dq}{d\tau}d\tau$$

$$=\int_0^t i(\tau)d\tau$$

$$\mathcal{L}\ [f*g]=\mathcal{L}\ [f]\mathcal{L}\ [g]$$

$$=\mathcal{L}\ \left[\int_0^t f(\tau)g(t-\tau)d\tau\right]$$

특별히 $g=1$

$$\mathcal{L}\ \left[\int_0^t f(\tau)d\tau\right]=\mathcal{L}\ [f]\frac{1}{s}$$

Ex)

$$L=0.2, R=2, C=0.4$$

$$i(0)=0$$

$$E(t)=\begin{cases}100t & t<1\\ 100 & t\geq 1\end{cases}$$

$$E=100t(u_0-u_1)+100u_1$$

$$y\left[0.2\frac{di}{dt}+2i(t)+\frac{1}{0.4}\int_0^t i(\tau)d\tau\right]=\mathcal{L}\ [E]$$

$$0.2(sI-i(0))+2I+\frac{1}{0.4}I\frac{1}{s}=100\frac{(1-e^{-s})}{s^2}$$

$$\mathcal{L}\ [E]=\mathcal{L}\ [u_0(t)100t]+[100(1-t)u_1]=$$

$$=100\frac{1}{s^2}-100\mathcal{L}\ [u_1(t-1)]$$

$$=\frac{100}{s^2}-100e^{-s}\frac{1}{s^2}$$

$$(0.2s+2+\frac{1}{0.4}\frac{1}{s})I=\frac{100}{s^2}(1-e^{-s})$$

$$(0.2s^2+2s+\frac{1}{0.4})I=\frac{100}{s}(1-e^{-s})$$

$$I=100(1-e^{-s})\frac{1}{s(0.2s^2+2s+\frac{5}{2})}$$

$$=100(1-e^{-s})\frac{2}{s(0.4s^2+4s+5)}$$

$$i(t)=(f(t)-u_1(t)f(t-1))$$

Ex)

$$L\frac{di}{dt}+Ri=E(t)$$

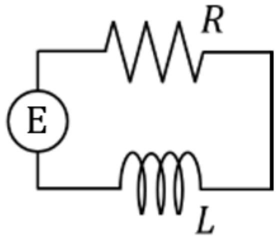

그림 11

$$LsI+RI=\frac{1}{s(1+e^{-s})}$$

$$\begin{aligned} I &= \frac{1}{(Ls+R)(s(1+e^{-s}))} \\ &= \frac{1}{s(Ls+R)}\sum_{k=0}^{\infty}(-1)^k e^{-ks} \end{aligned}$$

$$F(s)=\frac{1}{s(Ls+R)}$$

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s(L^2+R)}\right) &= \mathcal{L}^{-1}\left[\frac{L}{R}\left(\frac{1/L}{s}-\frac{1}{Ls+R}\right)\right] \\ &= \frac{L}{R}\left(\frac{1}{L}-\frac{1}{L}e^{-R/Lt}\right) \\ &= \frac{1}{R}\left(1-e^{-R/Lt}\right)=f(t) \end{aligned}$$

$$I(s)=\sum_{k=0}^{\infty}(-1)^k e^{-ks}F(s)$$

$$\begin{aligned} i(t) &= \sum_{k=0}^{\infty}(-1)^k \mathcal{L}^{-1}\left[e^{-ks}F(s)\right] \\ &= \sum_{k=0}^{\infty}(-1)^k u_k(t) f(t-k) \end{aligned}$$

$$0<t\le 1 \quad i(t)=f(t)$$
$$=\frac{1}{R}\left(1-e^{-R/Lt}\right)$$

$$1\le t<2 \quad i(t)=f(t)-u_1(t)f(t-1)$$

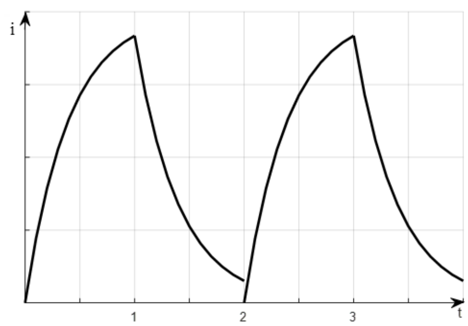

그림 12

$$
\begin{aligned}
i(t) &= \left(1-e^{-R/Lt}\right) = \left(1-e^{-R/L(t-1)}\right) \\
&= e^{-R/Lt}\left(e^{R/L}-1\right) > 0
\end{aligned}
$$

Ex)

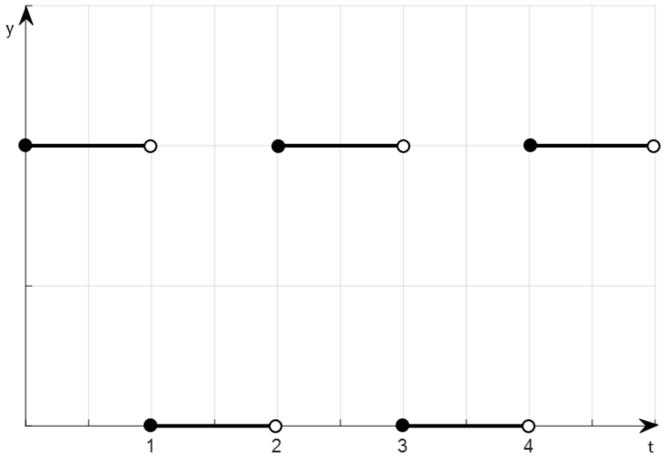

그림 13

$$f_k(t) = 1 - u_1(t) + u_2(t) - \cdots + (-1)^k u_k(t)$$

$$1 - u_1(t) = \begin{cases} 1 & t < 1 \\ 1 - 1 = 0 & t \geq 1 \end{cases}$$

$$f_k(a) = ?$$

$$
\begin{aligned}
u_j(a) &= 0 \qquad j > a \\
u_j(a) &= 1 \qquad j \leq a
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=1}^{k} (-1)^j u_k(a) &= \sum_{j \leq a} (-1)^j u_k(a) + \sum_{j > a} (-1)^j u_k(a) \\
&= \sum_{j \leq a} (-1)^j u_k(a) + 0
\end{aligned}
$$

$$\mathcal{L}[f] = \mathcal{L}[1] + \sum_{k=1}^{\infty} (-1)^k e^{-ks} \frac{1}{s} = \frac{1}{s} \frac{1}{1+e^{-s}}$$

$$
\begin{aligned}
\int_0^{\infty} e^{-st} f(t) dt &= \int_0^1 e^{-st} f(t) dt + \int_2^3 e^{-st} f(t) dt + \int_4^5 e^{-st} f(t) dt \\
&= \sum_{k=0}^{\infty} \int_{2k}^{2k+1} e^{-st} dt
\end{aligned}
$$

$$\mathcal{L}[u_{2k} - u_{2k+1}] = e^{-2ks} \frac{1}{s} - e^{-(2k+1)s} \frac{1}{s}$$

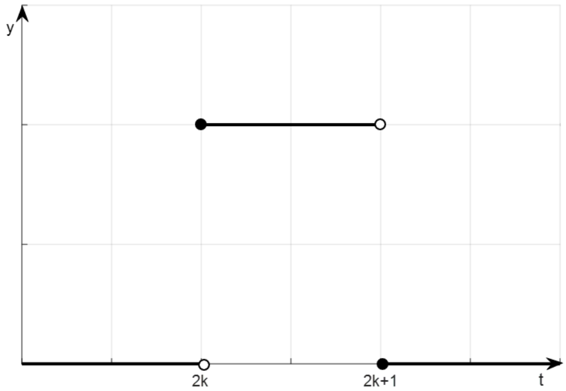

그림 14

$$\sum_{k=0}^{\infty}\frac{e^{-2ks}-e^{-(2k+1)s}}{s}=\frac{1-e^{-s}}{s}\sum_{k=0}^{\infty}e^{-2ks}$$
$$=\frac{1-e^{-s}}{s}\frac{1}{1-e^{-2s}}$$
$$|e^{-2s}|<1 \quad \text{If } s>0$$

Ex)

$$f(t+\tau)=f(k) \quad t \geq 0 \quad \tau>0$$
$$\mathcal{L}[f]=\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$$

Ex)

$$\begin{array}{c} f(k)=t \quad 0 \leq t<1 \\ f(t+1)=f(t) \end{array} \quad \text{Saw tooth function.}$$
$$\mathcal{L}[f]=\frac{\int_0^1 e^{-st}tdt}{1-e^{-s}}$$