Module Laplace Transform - 정의와 기본성질, 이동정리

1. Definition and basic properties

미분방정식의 해를 구하기 위해서 사용하는 변환법의 아이디어는 다음과 같다.

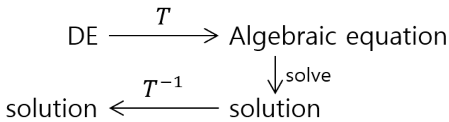


그림 1

미분방정식을 변환을 통해 대수방정식으로 변화하여 대수방정식의 해를 구하고 대수방정식의 해를 다시 역변환을 취하면 본래의 미분방정식의 해를 구할 수 있다.

왜 라플라스 변환인가?

다음과 같음 mass-spring system에서 시스템에 입력되는 함수가 불연속일 때 기존의 비동차 방정식의 풀이법을 사용할 수 없음.

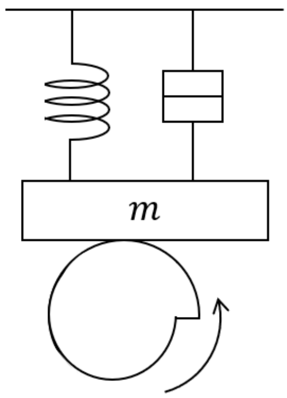


그림 2

라플라스변환의 정의

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt$$

Ex)

$$f = e^{at}$$

$$
\begin{aligned}
\mathcal{L}[e^{at}](s) &= \int_0^\infty e^{-st}e^{at}dt \\
&= \int_0^\infty e^{(a-s)t}dt \\
&= \lim_{k\to\infty}\int_0^k e^{(a-s)t}dt \\
&= \lim_{k\to\infty}\frac{1}{(a-s)}\left[e^{(a-s)k}-1\right] \\
&= \frac{1}{s-a} \qquad (a-s<0)
\end{aligned}
$$

Ex)

$$\mathcal{L}[\sin t](s) = \int_0^\infty e^{-st}\sin t\,dt$$

$$
\begin{aligned}
&\mathcal{L} = -e^{-st}\cos t|_0^\infty - s\int_0^\infty e^{-st}\cos t\,dt \\
&\quad = 1 - s(e^{-st}\sin t|_0^\infty + s\mathcal{L}) \\
&\mathcal{L} = 1 - s^2\mathcal{L} \\
&(1+s^2)\mathcal{L} = 1 \\
&\mathcal{L} = \frac{1}{1+s^2}
\end{aligned}
$$

Theorem.

$$
\begin{aligned}
&\mathcal{L}[\alpha f + \beta g] = \alpha F(s) + \beta G(s) \\
&F = \mathcal{L}[f],\ G = \mathcal{L}[g] \qquad s > a
\end{aligned}
$$

Q: When $\mathcal{L}[f](s)$ exist?

$f \mapsto \int_0^\infty e^{-st}f(t)dt$ Define for proper s.

Theorem.

Suppose f is piecewise continuous in $[0,k]$, for every positive k suppose $\exists\ M, b$ such that

$|f(t)| \le Me^{bt} \qquad t \ge 0$

$\Rightarrow \int_0^\infty e^{-st}f(t)dt$ Converges for $s > b$

Proof)

$\int_0^\infty e^{-st}|f(t)| \le M\int_0^\infty e^{(b-s)t}dt$ Finite when $b - s < 0$

Ex)

Even though $t^{-\frac{1}{2}}, t > 0$ does not satisfy growth condition.

$$\mathcal{L}\left[t^{-\frac{1}{2}}\right]=\int_0^\infty e^{-st}t^{-\frac{1}{2}}dt$$

$$\text{Let } x=t^{-\frac{1}{2}}, dx=\frac{1}{2}t^{-\frac{1}{2}}dt$$

$$\mathcal{L}[x]=2\int_0^\infty e^{-sx^2}dx \qquad \sqrt{5}\,x=y$$

$$=2\int_0^\infty e^{-y^2}\frac{1}{\sqrt{5}}dy$$

$$=\frac{2}{\sqrt{5}}\frac{\sqrt{\pi}}{2}=\sqrt{\frac{\pi}{5}}$$

Definition)

$$\mathcal{L}[g]=G, g=\mathcal{L}^{-1}[G]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right](t)=e^{at}$$

Ex)

$$\mathcal{L}[\cos(3t)-\sin(4t)](s)=\mathcal{L}[\cos(3t)](s)-\mathcal{L}[\sin(4t)](s)$$

$$=\frac{s}{s^2+3^2}-\frac{4}{s^2+4^2}$$

Ex)

$$\mathcal{L}\left[2t^2e^{-3t}-2t+1\right]=2\mathcal{L}\left[t^2e^{-3t}\right]-4\mathcal{L}[t]+\mathcal{L}[1]$$

$$=2\frac{2!}{(s+a)^3}-\frac{4}{s^2}+\frac{1}{s}$$

2. Inverse Laplace Transform

Ex)

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right](t)=e^{at}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t)=\sin t$$

Q: $\mathcal{L}^{-1}[g]$ is unique?

$$\mathcal{L}\left[e^{-t}\right](s)=\frac{1}{s+1},\quad s>-1$$

$$h(t)=\begin{cases} e^{-t} & t\neq 3 \\ 0 & t=3 \end{cases}$$

$$\mathcal{L}[h(t)](s)=\frac{1}{s+1}$$

$$\int_0^\infty e^{-st}h(t)dt=\int_0^3 e^{-st}h(t)dt+\int_3^\infty e^{-st}h(t)dt$$

$$=\int_0^\infty e^{-st}e^{-t}dt$$

ANS) If we demand $\mathcal{L}^{-1}[g]$ is continuous, then $\mathcal{L}^{-1}[g]$ is unique.

Ex)

$$\mathcal{L}^{-1}\left[\frac{2s-5}{s^2+16}\right]=2\mathcal{L}^{-1}\left[\frac{s}{s^2+4^2}\right]-5\mathcal{L}^{-1}\left[\frac{1}{s^2+4^2}\right]$$
$$=2\cos(4t)-\frac{5}{4}\sin(4t)$$

3. Differentiation and Laplace transform.

Essential theorem to use Laplace transform in solving differential equation.

Theorem 1.

Suppose that f is continuous and f' is piecewise continuous on $[0,A]$ for $A>0$.
Suppose that $\exists\ K,a,M>0$ such that $|f(t)|\le Ke^{at}$ for $t\ge M$
Then $\mathcal{L}[f']=s\mathcal{L}[f]-f(0)$

Proof)

$$①\int_0^A e^{-st}f'(t)dt=\int_0^{t_1}e^{-st}f'(t)dt+\int_{t_1}^{t_2}+\cdots\int_{t_k}^{A}e^{-st}f'(t)dt$$
$$t_1,t_2,\cdots,t_k\in(0,A)$$

Points of discontinuity of f'

$$\int_{t_{i-1}}^{t_i}e^{-st}f'(t)dt=e^{-st}f(t)|_{t_{i-1}}^{t_i}=-e^{-s0}f(o)+e^{-st}f(A)$$
$$\lim_{A\to\infty}|e^{-sA}f(A)|\le\lim_{A\to\infty}e^{-sA}ke^{aA}\quad e^{-(s-a)A}\to 0,\ \ s>a$$

Corollary 2.

$f,\cdots,f^{(n-1)}$ continuous $f^{(n)}$ piecewise continuous on $[0,A]$
$\exists\ K,a,M>0$ Such that $|f^{(i)}(t)|\le ke^{at}$ for $t\ge M,\ i=0,1,2,\cdots,n-1$

$$\Rightarrow\mathcal{L}[f^{(n)}]=s^n\mathcal{L}[f]-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)})(0)$$

Table 1.

f	$\mathcal{L}[f](s)$
1	$\frac{1}{s}\quad s>0$
t^n $n=1,2,\cdots$	$\frac{n!}{s^{n+1}}\quad s>0$
$\cos(kt)$	$\frac{s}{s^2+k^2}\quad s>0$

$\sin(kt)$	$\dfrac{k}{s^2+k^2} \quad s>0$
$\vdots$	$\vdots$
e^{at}	$\dfrac{1}{s-a} \quad s>0$

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'] - f'(0)\\ &= s[sF(s)-f(0)]-f'(0)\\ &= s^2 f(s) - sf(0) - f'(0)\end{aligned}$$

Ex)

$$y' - 4y = 1, \quad y(0) = 1$$
$$\mathcal{L}[y'-4y] = \mathcal{L}[1]$$
$$sY(s) - y(0) - 4Y(s) = \frac{1}{s}$$
$$(s-4)Y(s) = 1 + \frac{1}{s}$$
$$\Rightarrow \; Y(s) = \frac{s+1}{s(s-4)}$$
$$= \frac{1}{s(s-4)} + \frac{1}{s-4}$$
$$\mathcal{L}[y] = Y(s)$$
$$\begin{aligned}y(t) &= \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{1}{s(s-4)} + \frac{1}{s-4}\right]\\ &= \frac{1}{4-0}(e^{4t}-1) + e^{4t}\\ &= \frac{5}{4}e^{4t} - \frac{1}{4}\end{aligned}$$

Ex)

$$y'' + 4y' + 3y = e^t \qquad y(0)=0, y'(0)=2$$
$$\mathcal{L}[y''+4y'+3y] = \mathcal{L}[e^t]$$
$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 3Y(s) = \frac{1}{s-a}$$
$$s^2Y - 2 + 4sY + 3Y = \frac{1}{s-1}$$
$$(s^2+4s+3)Y = 2 + \frac{1}{s-1} = \frac{2s-1}{s-1}$$
$$Y = \frac{2}{s^2+4s+3} + \frac{1}{(s-1)(s^2+4s+3}$$
$$\begin{aligned}Y &= \frac{2s-1}{(s-1)(s+1)(s+3)}\\ &= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}\end{aligned}$$
$$2s-1 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$$

$$\begin{cases} s=-1 & -3=(-2)(2)B & B=3/4 \\ s=1 & 1=(2)(4)A & A=1/8 \\ s=-3 & -7=(-4)(-2)C & C= 7/8 \end{cases}$$

$$Y=\frac{1/8}{s-1}+\frac{3/4}{s+1}-\frac{7/8}{s+3}$$

$$\begin{aligned} y &= \mathcal{L}^{-1}[Y] \\ &= \frac{1}{8}e^{t}+\frac{3}{4}e^{-t}-\frac{7}{8}e^{-3t} \end{aligned}$$

4. Shifting theorem. (이동정리)

1) 왜 shifting theorem이 필요한가?

비동차 이계선형 미분방정식에 라플라스 변환을 적용했을 때 맞이하는 상황 =>

$$s^2Y(s)-sy(0)-y'(0)+A(sY(s)-y(0))+BY(s)=(s^2+As+B)Y(s)-y(0)s-y'(0)-Ay(0)$$

$$(s^2+As+B)Y(s)=y_1+F(s)$$

$$\begin{aligned} Y(s) &= \frac{y_1}{s^2+As+B}+F(s)\frac{1}{s^2+As+B} \\ &= \frac{y_1}{(s-\alpha)(s-\beta)} \end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{1}{\alpha-\beta}\left(\frac{1}{s-\alpha}-\frac{1}{s-\beta}\right)\right]=\frac{1}{\alpha-\beta}(e^{\alpha t}-e^{\beta t})$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-\alpha)^2}\right]=?$$

$$\mathcal{L}[t]=\frac{1}{2^2}$$

$$\mathcal{L}[e^{\alpha t}t]=\frac{1}{(s-\alpha)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\alpha)^2+\beta^2}\right]=?$$

2) Step functions. (shifting theorems)

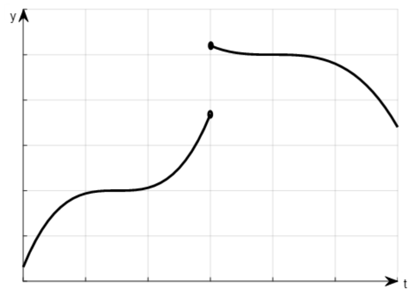
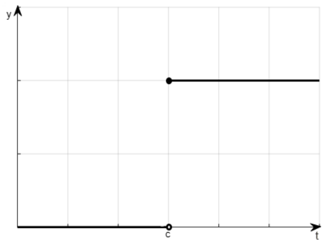


그림 3

불연속 함수 f의 라플라스 변환 $\mathcal{L}[f]=?$

$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

Step function / Heaviside function



Rectangular pulse.

그림 4

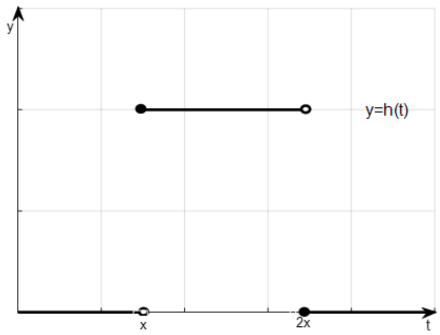


그림 5

$$h(t) = \begin{cases} 0 & t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

$$h(t) = t_\pi(t) - u_{2\pi}(t)$$

$$\begin{aligned} \mathcal{L}[u_c(t)] &= \int_0^\infty e^{-st} u_c(t) dt \\ &= \int_c^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_c^\infty \\ &= \frac{1}{s} e^{-cs}, \quad s > 0 \end{aligned}$$

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

$$g = u_c(t) f(t-c)$$

$$\mathcal{L}[g] = ?$$

Theorem. (이동정리)

$$\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} \mathcal{L}[f]$$

Proof)

$$\int_0^\infty e^{-st}u_c(t)f(t-c)dt$$
$$=\int_0^\infty e^{-st}f(t-c)dt \qquad \tau=t-c$$
$$=\int_0^\infty e^{-s(\tau-c)}f(\tau)d\tau$$
$$=e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau= e^{-cs}\mathcal{L}[f](s)$$

Ex)

$$f(t)=\begin{cases}\sin t & 0\le t<\frac{\pi}{4}\\ \sin t+\cos(t-\frac{\pi}{4}) & t\ge\frac{\pi}{4}\end{cases}$$

$$\mathcal{L}[f]=?$$

$$f(t)=\sin t+u_{\pi/4}\cos(t-\frac{\pi}{4})$$

$$\mathcal{L}[f]=\mathcal{L}[\sin t]+\mathcal{L}[u_{\pi/4}(t)\cos(t-\pi/4)]$$
$$=\frac{1}{s^2+1}+e^{-\pi/4s}\mathcal{L}[\cos t]$$
$$=\frac{1}{s^2+1}+e^{-\pi/4s}\frac{s}{s^2+1}$$

Ex)

$$F(s)=\frac{1-e^{-2s}}{s^2}$$

$$\mathcal{L}^{-1}[F]=?$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]-\mathcal{L}\left[e^{-2s}\frac{1}{s^2}\right]$$

$$\mathcal{L}[t]=\frac{1}{s^2}, \qquad \mathcal{L}[t^n]=\frac{n!}{s^{n+1}}(s>0)$$

$$\mathcal{L}[u_2(t)(t-2)]=e^{-2s}\frac{1}{s^2}$$

ANS)

$$t-u_2(t)(t-2)=\begin{cases}t & t<2\\ 2 & t\ge 2\end{cases}$$

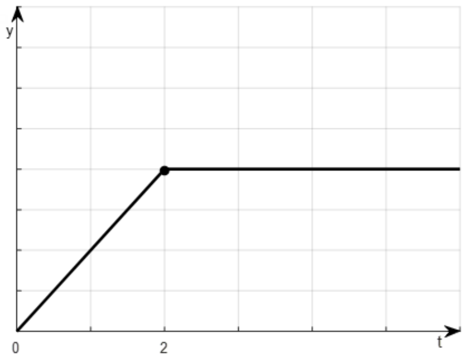


그림 6

Motivation: inverse Laplace transformation of shifted function

Theorem.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c), \quad s < a+c$$

$$\text{if } \mathcal{L}[f](s) = F(s)$$

Ex)

$$\mathcal{L}[te^{2t}] = \mathcal{L}[t](s-2) = \frac{1}{(s-2)^2}$$

Ex)

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-4s+5}\right]$$

$$s^2-4s+5 = (s-2)^2+1$$

$$D = (-4)^2 - 4\times 5 = 16-20 < 0$$

$$\frac{1}{(s-2)^2+1} \leftarrow \frac{1}{s^2+1}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$\mathcal{L}\left[e^{ct}\sin t\right] = \frac{1}{(s-c)^2+1}$$

$$\Rightarrow c = 2$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}\sin t$$