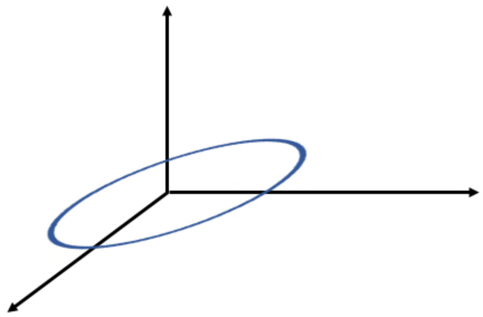
## Module Surface integral and flux

### 1. Parametrizations of Surfaces: Introduction

①



$$x = \cos t$$

$$y = \sin t$$

$$z = 0$$

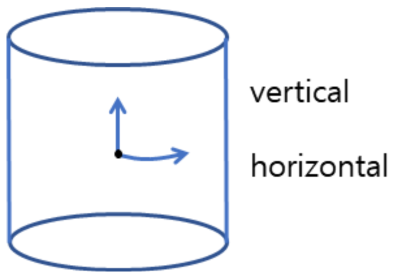
We want to express the ant circle on plane $z = 3$

$$x = \cos t$$

$$y = \sin t$$

$$z = 3$$

We need two parameters $t$ and $s$ such that we express points moving on cylinder.




$$R^2 \ni (s,t) \mapsto (x(s,t), y(s,t), z(s,t)) \in R^3$$

② $z = f(x,y)$ paramerize a surface given as a graph of $f$

$$x = t, y = s \qquad z = f(t,s)$$

예제)

$$x^2 + y^2 + z^2 = 1, z \geqq 0$$

$$z = \sqrt{1 - x^2 - y^2}$$

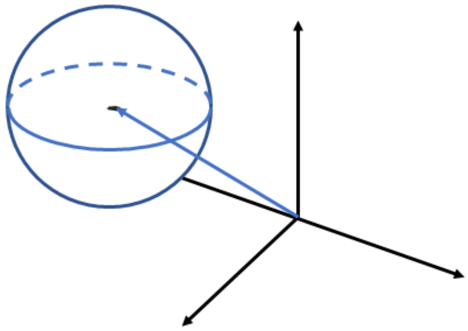
$$x = t, y = s, z = \sqrt{1 - t^2 - s^2}$$

$$(t,s)\in\{t^2+s^2\leqq 1\}$$

③ parametrization of plane.

$$p=p_0=tV+sW$$

예제) Sphere centered at $(2,-1,3)$ with radius 2

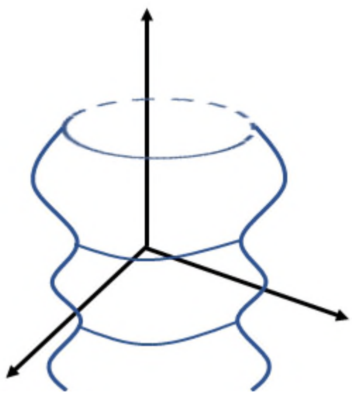


$$\vec{r}(\varphi,\theta)=\vec{p}+\vec{r_0}(\varphi,\theta)$$

$$=(2,-1,3)+2(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi)$$

$$\begin{cases} x=2+2\sin\varphi\cos\theta & 0\leqq \varphi\leqq \pi \\ y=-1+2\sin\varphi\sin\theta & 0\leqq \theta\leqq 2\pi \\ z=3+2\cos\varphi & \end{cases}$$

④ Surface of revolution



curve in x-z plane

$$x=x_0(s)$$

$$y=0$$

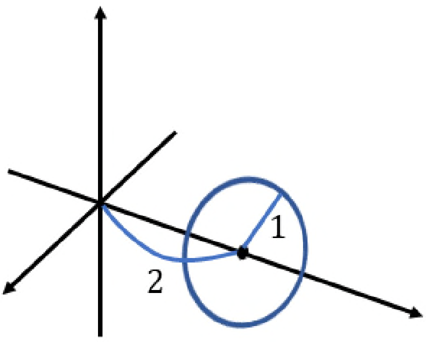
$$z=z_0(s)$$

$$x(t,s) = x_0(s)\cos t \qquad 0 \leqq t \leqq 2\pi$$

$$y(t,s) = y_0(s)\sin t \qquad a \leqq s \leqq b$$

$$z(t,s) = z_0(s)$$

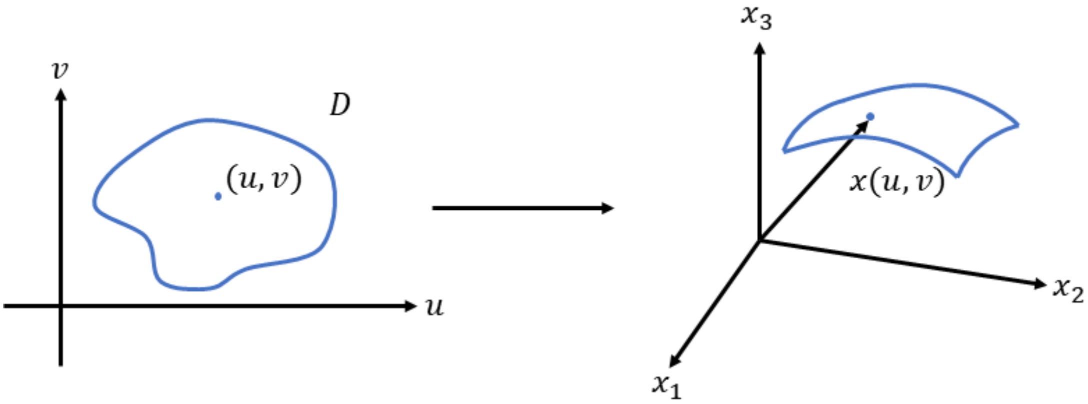
예제) torus




$$x_0 = 2 + \cos(s) \qquad 0 \leqq s \leqq 2\pi$$

$$z_0 = 2 + \sin(s)$$

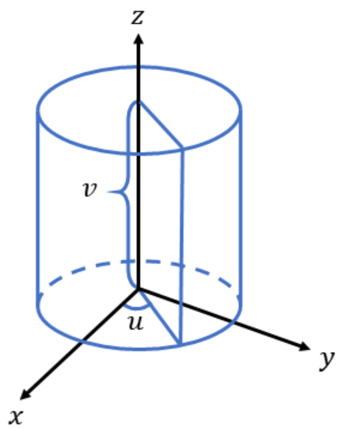
## 2. 곡면의 매개화




$$X : D \to R^3 \text{ 연속 사상 (mapping)}$$

$$(u,v) \mapsto X(u,v) = (X_1(u,v), X_2(u,v), X_3(u,v))$$

예제)

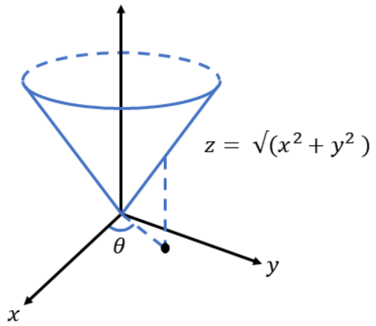



$x^2 + y^2 = 4$ $\qquad x(u,v) = 2\cos u \quad (u,v) \in [0,2\pi) \times R$

$y(u,v) = 2\sin u$

$z(u,v) = v$

예제)

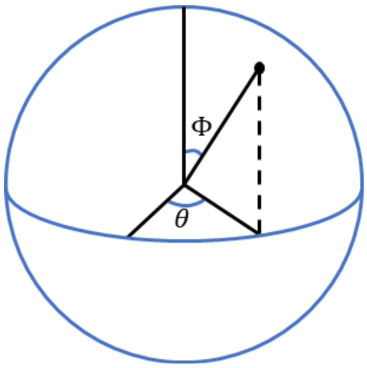


$x = r\cos\theta \qquad (r,\theta) \in R^{+} \times [0,2\pi)$

$y = r\sin\theta$

$z = \sqrt{x^2 + y^2} = r$

예제) sphere




$$x^2 + y^2 + z^2 = 1$$

$$x = \sin\phi\cos\theta$$

$$x = \sin\phi\sin\theta$$

$$z = \cos\phi$$

$$(\phi,\theta) \in [0,\pi] \times [0,2\pi)$$

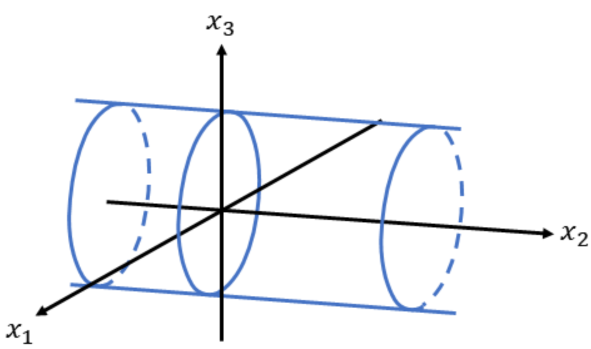
예제) 매개화가 주어졌을 때 곡면을 sketch 해보라.

$$X(u,v) = (2\cos u, v, 2\sin u)$$

$x_1 = 2\cos u \qquad x_1^2 + x_3^2 = (2)^2$

$x_2 = v$

$x_3 = 2\sin u$



$x_1 - x_3$평면에 중심이(0,0)이고 반지름이 2인 원. $x_2$방향으로 이동

예제) $X(u,v) = (u\cos v, u\sin v, u^2)$으로 매개화 되는 곡면을 스케치하라.

## 3. 곡면의 면적 구하기

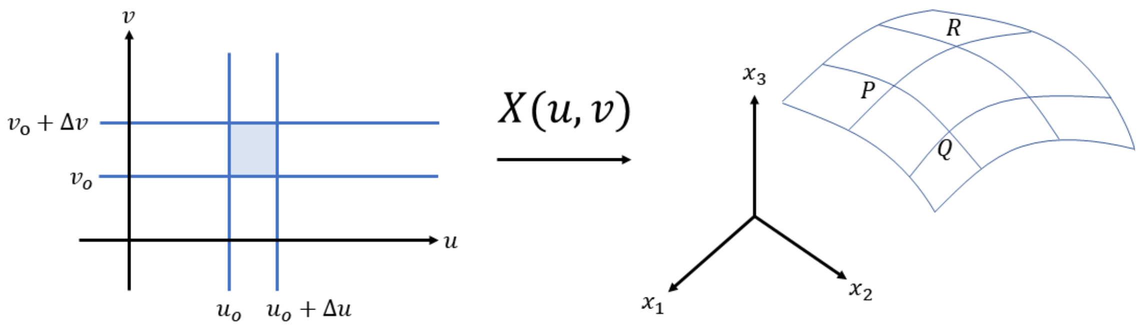
$$S: \quad X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)) \quad (u,v) \in D$$

$$A(S) = \iint_D |X_u \times X_v| dudv$$

$$X_u = \frac{\partial x_1}{\partial u} i + \frac{\partial x_2}{\partial u} j + \frac{\partial x_3}{\partial u} k$$

$$X_v = \frac{\partial x_1}{\partial v} i + \frac{\partial x_2}{\partial v} j + \frac{\partial x_3}{\partial v} k$$

Justification of area formula




(*) $[u_0, u_0 + \triangle u] \times [v_0, v_0 + \triangle v]$의 $X$에 의한 image의 면적을 구해보자.

$$\text{arc PQ의 길이} = \begin{pmatrix} curve & u \mapsto x(u, v_0) \\ & u_0 \leq u \leq u_0 + \triangle u \end{pmatrix} \text{의 길이}$$

$$\text{arc PR의 길이} = \begin{pmatrix} curve & v \mapsto x(u_0, v) \\ & v_0 \leq v \leq v_0 + \triangle v \end{pmatrix} \text{의 길이}$$

(*) 면적 ≈ $|\overrightarrow{PQ} \times \overrightarrow{PR}|$

curve $u \mapsto x(u, v_0)$의 $u = u_0$에서의 target vector를 $X_u$라고 놓자.

$$u \mapsto (x_1(u, v_0), x_2(u, v_0), x_3(u, v_0))$$

$$X_u|_{(u_0, v_0)} = \left( \frac{\partial x_1}{\partial u}(u_0, v_0), \frac{\partial x_2}{\partial u}(u_0, v_0), \frac{\partial x_3}{\partial u}(u_0, v_0) \right)$$

curve $v \mapsto X(u_0, v)$의 $v = v_0$에서의 target vector를 $X_v$라고 놓자.

$$v \mapsto (x_1(u_0, v), x_2(u_0, v), x_3(u_0, v))$$

$$X_v|_{(u_0, v_0)} = \left( \frac{\partial x_1}{\partial v}(u_0, v_0), \frac{\partial x_2}{\partial v}(u_0, v_0), \frac{\partial x_3}{\partial v}(u_0, v_0) \right)$$

$$\overrightarrow{PQ} \approx X_u\big|_{(u_0,v_0)} \triangle u$$

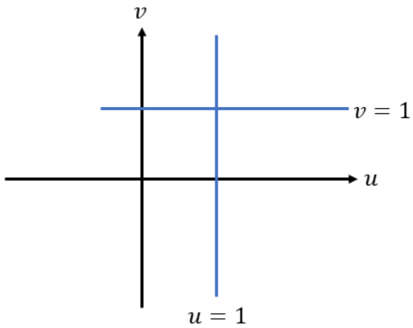
$$\overrightarrow{PR} \approx X_v\big|_{(u_0,v_0)} \triangle v$$

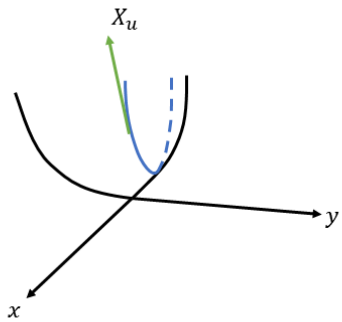
$$|\overrightarrow{PQ} \times \overrightarrow{PR}| \approx |X_u \times X_v| \triangle u \triangle v$$

$$\text{surface의 면적} \approx \sum\sum |X_u \times X_v| \triangle u \triangle v$$

예제)

$$x = u, y = v, z = u^2 + v^2$$




$$u = 1 \quad \Rightarrow \quad (1, v, 1 + v^2)$$

$$v = 1 \quad \Rightarrow \quad (u, 1, 1 + u^2)$$

$$X(u,v) = (u, v, u^2 + v^2)$$

$$X_u = \left(\frac{\partial u}{\partial u}, \frac{\partial v}{\partial u}, \frac{\partial u^2 + v^2}{\partial u}\right)$$

$$= (1, 0, 2u)$$

$$X_v = \left(\frac{\partial u}{\partial v}, \frac{\partial v}{\partial v}, \frac{\partial u^2 + v^2}{\partial v}\right) = (0, 1, 2v)$$

예제) 반지름 1인 sphere 의 겉면적 구하기

$$x_1 = \sin\phi\cos\theta \qquad x_2 = \sin\phi\sin\theta \qquad x_3 = \cos\phi$$

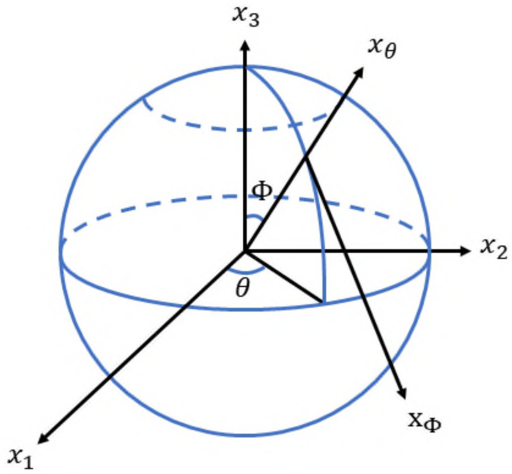
$$D = \{(\phi, 0) : 0 \leqq \phi \leqq \pi, 0 \leqq \theta \leqq 2\pi\}$$

$$X_\phi = \frac{\partial x_1}{\partial \phi} i + \frac{\partial x_2}{\partial \phi} j + \frac{\partial x_3}{\partial \phi} k$$

$$= \cos\phi\cos\theta i + \cos\phi\sin\theta j - \sin\phi k$$

$$X_\phi = \frac{\partial x_1}{\partial \theta} i + \frac{\partial x_2}{\partial \theta} j + \frac{\partial x_3}{\partial \theta} k$$

$$= -\sin\phi \sin\theta i + \sin\phi \cos\theta j + 0k$$




$$X_\phi \times X_\theta = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\phi\cos\theta i + \sin^2\phi\sin\theta j + \sin\phi\cos\phi k$$

$$|X_\phi \times X_\theta|^2 = \sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi$$

$$= \sin^4\phi + \sin^2\phi\cos^2\phi$$

$$= \sin^2\phi(\sin^2\phi + \cos^2\phi)$$

$$= \sin^2\phi$$

$$A(S) = \iint_D |X_\phi \times X_\theta| d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi d\phi d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi d\phi = 4\pi$$

예제) Surface가 함수 $z = f(x,y)$의 그래프로 주어질 때, $S: z = f(x,y)$, $(x,y) \in D$ $S$의 surface area가 $\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dxdy$로 주어짐을 증명하라.

예제) 평면 $z = 9$ 아래에 있는 paraboloid $z = x^2 + y^2$의 surface area를 위의 결과를 이용하여 계산하라.

예제) The portion of the cone $z=\sqrt{x^2+y^2}$ between the planes $z=2$ and $z=6$
Find its surface area.

예제) Cylindrical surface $y^2+z^2=4$ $1\leqq x\leqq 4$

$$\rho(x,y,z)=z^2$$

$$\vec{r}(u,v)=(v,2\cos u,2\sin u) \qquad 0\leqq u\leqq 2\pi,\ 1\leqq v\leqq 4$$

$$\vec{r_u}\times\vec{r_v}=2\cos u\vec{j}+2\sin u\vec{k}$$

$$\text{mass} = \int_1^4\int_0^{2\pi}(2\sin u)^2 2dudv$$

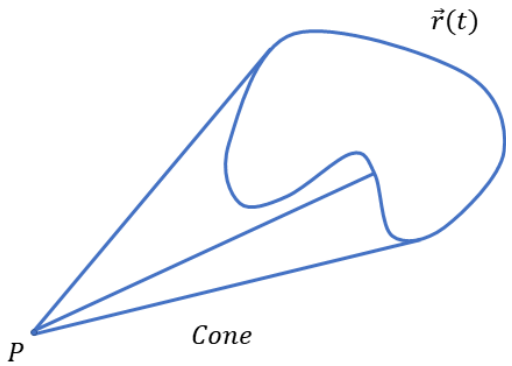
$$=8\int_1^4 dv\int_0^{2\pi}\sin^2 udu$$

예제) $z\geqq 0$, $\rho=z$ ⇒ $0\leqq u\leqq \pi$

$$\int_1^4\int_0^{\pi}(2\sin u)2dudv$$

예제) Torus의 surface area?

예제)




$$\vec{r}(t,s)=(1-s)\vec{p}+s\vec{r}(t) \qquad 0\leqq s\leqq 1\ ,\ a\leqq t\leqq b$$

$$\vec{r_t}=s\vec{r}'(t)$$

$$\vec{r_s}=-\vec{p}+\vec{r}(t)$$

$$|\vec{r_t}\times\vec{r_s}|=s|\vec{r}'(t)\times(-\vec{p}+\vec{r}(t))|$$

예제)

$$P=(0,0,0) \qquad \vec{r}(t)=(3,5\cos t,10\sin t) \qquad 0 \leqq t \leqq 2\pi$$

$$\vec{r_t}\times\vec{r_s}=s\begin{vmatrix} i & j & k \\ 0 & -5\sin t & 10\cos t \\ 3 & 5\cos t & 10\sin t \end{vmatrix}$$

$$=s\left[(-50)\vec{i}+30\cos t\vec{j}+15\sin t\vec{k}\right]$$

$$\text{Area} = \int_0^1\int_0^{2\pi} s\sqrt{50^2+30^2\cos^2 t+15^2\sin^2 t}\,dtds$$

## 4. Surface integral of f over the surface S

$$S: X: D\rightarrow R^3$$

$$\iint_S f d\sigma=\iint_D f(X(u,v))|X_u\times X_v|dudv$$

A thin sheet has the shape of a surface S and its density (mass per unit area) at $(x,y,z)$ is $\rho(x,y,z)$

Total mass of the sheet

$$m=\iint_S \rho d\sigma$$

Center of mass $(\bar{x},\bar{y},\bar{z})$

$$\bar{x}=\frac{1}{m}\iint_S x\rho d\sigma$$

$$\bar{y}=\frac{1}{m}\iint_S y\rho d\sigma$$

$$\bar{z}=\frac{1}{m}\iint_S z\rho d\sigma$$

예제) $S: z=\sqrt{a^2-x^2-y^2}\ \ (x^2+y^2\leqq a^2), (x,y,z)=z.$ Find $(\bar{x},\bar{y},\bar{z})$.

Sol) S를 매개화한다.

$$X(u,v)=(u,v,\sqrt{a^2-u^2-v^2})$$

$$D=\{(u,v): u^2+v^2\leqq a^2\}$$

$$X_u=\left(1,0,\frac{\partial z}{\partial u}\right)$$

$$X_v=\left(0,1,\frac{\partial z}{\partial v}\right)$$

$$z^2 = a^2 - u^2 - v^2$$

$$u^2 + v^2 + z^2 = a^2$$

Apply $\frac{\partial}{\partial u}$ to both sides $2u + 2z\frac{\partial z}{\partial u} = 0 \quad \frac{\partial z}{\partial u} = -\frac{u}{z}$

Like wise $2v + 2z\frac{\partial z}{\partial v} = 0 \quad \frac{\partial z}{\partial v} = -\frac{v}{z}$

$$X_u = \left(1, 0, -\frac{u}{z}\right) = \left(1, 0, -\frac{u}{\sqrt{a^2 - u^2 - v^2}}\right)$$

$$X_v = \left(0, 1, -\frac{v}{z}\right) = \left(0, 1, -\frac{v}{\sqrt{a^2 - u^2 - v^2}}\right)$$

$$|X_u \times X_v| = \left\| \begin{matrix} i & j & k \\ 1 & 0 & -\frac{u}{z} \\ 0 & 1 & -\frac{v}{z} \end{matrix} \right\|$$

$$= \left| \frac{u}{z}i + \frac{v}{z}j + k \right|$$

$$= \sqrt{\left(\frac{u}{z}\right)^2 + \left(\frac{v}{z}\right)^2 + 1}$$

$$= \frac{a}{z} = \frac{a}{\sqrt{a^2 - u^2 - v^2}}$$

$$m = \iint_S \rho d\sigma = \iint_D \rho(u, v, z(u, v)) \frac{a}{\sqrt{a^2 - u^2 - v^2}} dudv$$

use polar coordinates

$$= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \frac{a}{\sqrt{a^2 - r^2}} rdrd\theta$$

$$= 2\pi a \frac{1}{2} a^2 = \pi a^3$$

$$\bar{x} = \frac{1}{m} \iint_S x \rho d\sigma$$

$$\iint_S x \rho d\sigma = \iint_D u \rho(u, v, z(u, v)) \frac{a}{\sqrt{a^2 - u^2 - v^2}} dudv \ \text{(여기서 } x(u,v) = u\text{)}$$

$$= \int_0^{2\pi} \int_0^a r\cos\theta ardrd\theta$$

$$= a \int_0^a r^2 dr \int_0^{2\pi} \cos\theta d\theta = 0$$

Likewise $\bar{y}=0,\ \bar{z}=\frac{2}{3}a$

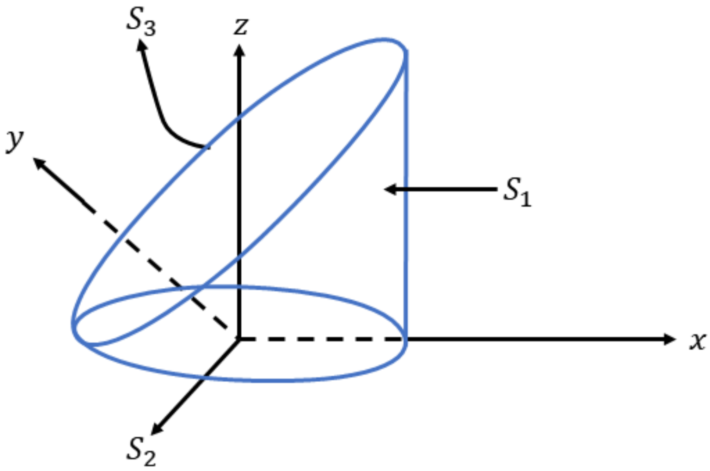
예제)

$$S=S_1\cup\ S_2\cup\ S_3$$

$$S_1:\{x^2+y^2=1\}\cap\ \{0\leqq\ z\leqq\ 1+x\}$$

$$S_2:\{z=0\}\cap\ \{x^2+y^2\leqq\ 1\}$$

$$S_3:\{z=1+x\}\cap\ \{x^2+y^2\leqq\ 1\}$$




$$\iint_S zd\sigma=\iint_{S_1} zd\sigma+\iint_{S_2} zd\sigma+\iint_{S_3} zd\sigma$$

$$S_1:x=\cos\theta, y=\sin\theta, z=z$$

$$D=\{0\leqq\ \theta\leqq\ 2\pi, 0\leqq\ z\leqq\ 1+\cos\theta\}$$

$$X(\theta,z)=(\cos\theta,\sin\theta,z)$$

$$X_\theta=(-\sin\theta,\cos\theta,0)$$

$$X_z=(0,0,1)$$

$$X_\theta\times X_z=\begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=\cos\theta i+\sin\theta j$$

$$|X_\theta\times X_z|=1$$

$$\iint_{S_1} z d\sigma = \iint_D z dz d\theta$$

$$= \int_0^{2\pi} \int_0^{1+\cos\theta} z dz d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta = \frac{3\pi}{2}$$

$$\iint_{S_2} z d\sigma = 0 \ (z=0 \ on \ S_2)$$

$$\iint_{S_3} z d\sigma = \sqrt{2}\pi \ \ \text{(Exe)}$$

Exe)

$S: x+y+z=1,\ x \geq 0, y \geq 0, z \geq 0$

$\iint_S yz d\sigma$ 를 계산하라. (ans : $\frac{\sqrt{3}}{24}$)

## 5. Flux integral through surface

*Orientation of surface

곡면의 안과 밖을 구분하는 방법이다.

곡면의 법선 vector를 이용하여 orientation 을 정한다.

예제) $x^2+y^2+z^2=1$

바깥 쪽을 향하는 법선 vector는 positive orientation 을 안쪽을 향하는 법선 vector는 negative orientation을 준다.

$$X(\phi,\theta) = \sin\phi\cos\theta i + \sin\phi\sin\theta j + \cos\phi k$$

$$X_\phi \times X_\theta = \sin^2\phi\cos\theta i + \sin^2\phi\sin\theta j + \sin\phi\cos\phi k$$

--------------------------------------------

*곡면 위에서의 적분의 중요한 응용으로 Flux integral을 생각해보자.

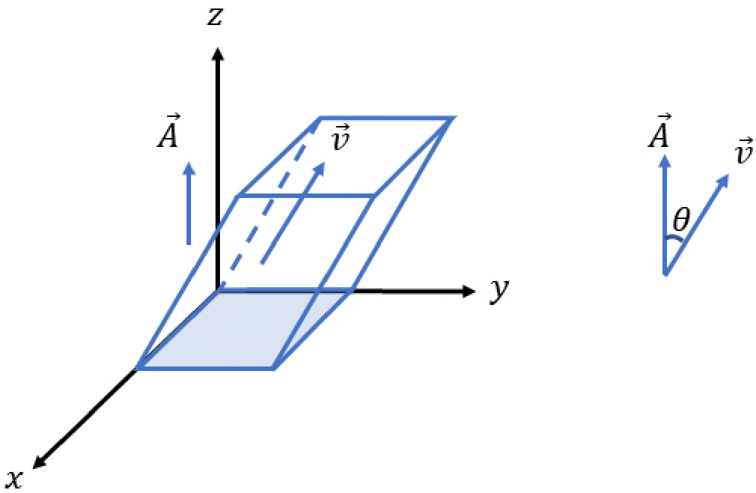
Flux = 단위시간 동안 S를 통과한 물의 volume (부피/시간)

Def) Area vector $\vec{A}$

i ) $|\vec{A}|$ = area of the surface

ii ) $\vec{A} // \vec{n}$

예제)

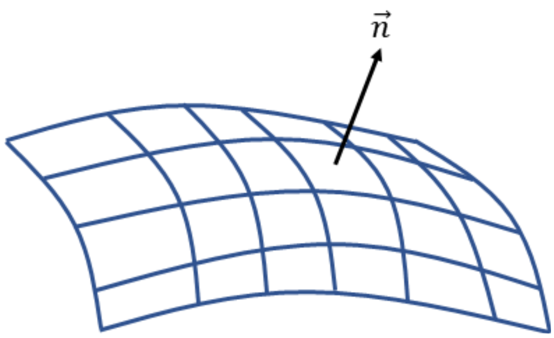



$$\overrightarrow{A} = A\overrightarrow{k} \qquad \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{k}$$

A는 공간의 면적

$\triangle t$ 동안 $S$를 통과한 물의 부피 $= |\overrightarrow{A}||\overrightarrow{v}|\triangle t \cos\theta$

Flux $= |\overrightarrow{A}||\overrightarrow{v}|\cos\theta = \overrightarrow{A} \cdot \overrightarrow{v}$




$$\triangle\overrightarrow{A} = \triangle A\overrightarrow{n}$$

$\sum F \cdot \triangle\overrightarrow{A} =$ Flux through

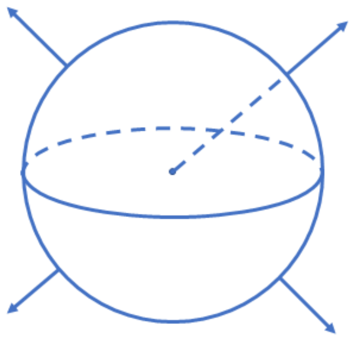
$$\rightarrow \iint_S F \cdot n d\sigma$$

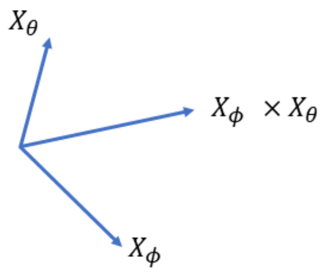
$$|X_\phi \times X_\theta| = \sin\phi$$

$$n = \frac{X_\phi \times X_\theta}{|X_\phi \times X_\theta|} = \sin\phi\cos\theta i + \sin\phi\sin\theta j + \cos\phi k$$

normal vector (법선 벡터)

$n =$ sphere의 position vector



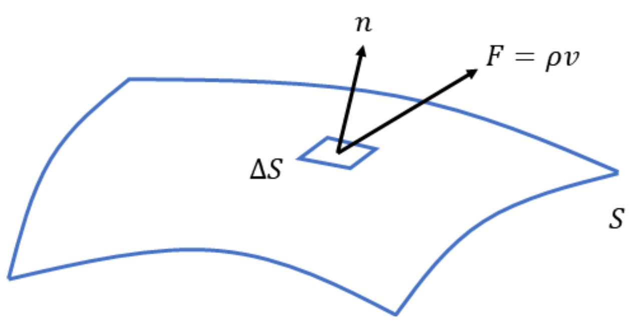



**(정의) Flux integral of vector field across S**

F는 S위에서 정의된 연속하는 vector field.

$n : S$의 outward normal vector

$$\iint_S F \cdot n d\sigma$$




Flux integral의 물리적 의미

density $\rho(x,y,z)$, velocity $v(x,y,z)$인 fluid의 flow를 생각하라.

$(\rho v \cdot n)$ Area $(\Delta S)$는 surface 조각 $\Delta S$를 단위시간당 통과하는 유체의 양이다.

$\sum_S (\rho v \cdot n) Area(\Delta S)$는 단위시간당 $S$를 통과하는 유체의 양이다.

예제)

$F = zi + yj + xk$

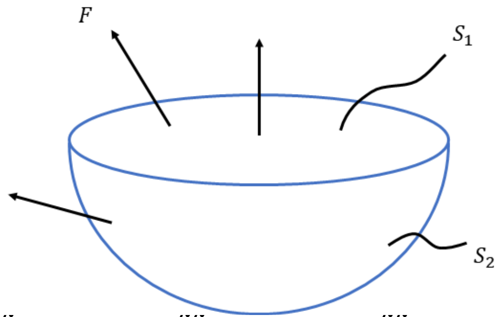
$S : x^2 + y^2 + z^2 = 1$

$\iint_S F\cdot\ nd\sigma$를 계산해보자.

$$X(\phi,\theta)=(\sin\phi,\cos\theta,\sin\phi,\sin\theta,\cos\phi)$$
$$D=\{(\phi,\theta):0\leqq\ \phi\leqq\ \pi,0\leqq\ \theta\leqq\ 2\pi\}$$
$$\iint_D F(X(\phi,\theta))\cdot\ \frac{X_\phi\times X_\theta}{|X_\phi\times X_\theta|}|X_\phi\times X_\theta|d\phi d\theta$$
$$F(X(\phi,\theta))=\cos\phi i+\sin\phi\sin\theta j+\sin\phi\cos\theta k$$
$$X_\phi\times X_\theta=\sin^2\phi\cos\theta i+\sin^2\phi\sin\theta j+\sin\phi\cos\phi k$$
$$F(X(\phi,\theta))\cdot\ (X_\phi\times X_\theta)$$
$$=\cos\phi\sin^2\phi\cos\theta+\sin^3\phi\sin^2\theta+\sin^2\phi\cos\phi\cos\theta$$
$$=2\cos\phi\sin^2\phi\cos\theta+\sin^3\phi\sin^2\theta$$
$$\int_0^{2\pi}\int_0^{\pi}(2\cos\phi\sin^2\phi\cos\theta+\sin^3\phi\sin^2\theta)d\phi d\theta$$
$$=2\int_0^{2\pi}\cos\theta d\theta\int_0^{\pi}\cos\phi\sin^2\phi d\phi+\int_0^{2\pi}\sin^2\theta d\theta\int_0^{\pi}\sin^3\phi d\phi$$
$$=0+\frac{4\pi}{3}=\frac{4\pi}{3}$$

예제)

$$F=xi+yj+zk$$
$$S:\{x^2+y^2\leqq\ z\leqq\ 4\}$$




$$\iint_S F\cdot\ nd\sigma=\iint_{S_1}F\cdot\ n_1d\sigma+\iint_{S_2}F\cdot\ n_2d\sigma$$
$$S_1:n_1=k$$
$$X(u,v)=(u,v,4)$$
$$D=\{(u,v):u^2+v^2\leqq\ 4\}$$
$$F(X(u,v))\cdot\ n_1=4$$
$$X_u=(1,0,0),\ X_v=(0,1,0)$$

$$X_u \times X_v = k$$

$$\iint_{S_1} F \cdot \ n_1 = \iint_D 4dudv$$

$$= 4Area(D) = 4 \cdot \ \pi(2)^2 = 16\pi$$

$$S_2 : X(u,v) = (u,v,u^2+v^2)$$

$$X_u = (1,0,2u)$$

$$X_v = (0,1,2v)$$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

$$= -2ui - 2vj + k$$

$$\iint_D F \cdot \ -\frac{(X_u \times X_v)}{|X_u \times X_v|} |X_u \times X_v| dudv$$

$$= \iint_D F \cdot \ (-X_u \times X_v) dudv$$

$$F(X(u,v)) = ui + vj + (u^2+v^2)k$$

$$F \cdot \ (-X_u \times X_v) = 2u^2 + 2v^2 - u^2 + v^2$$

$$= u^2 + v^2$$

$$\iint_D (u^2+v^2) dudv$$

$$\int_0^{2\pi} \int_0^2 r^3 dr d\theta = 8\pi$$

$$\therefore \ \iint_S F \cdot \ ndr = 16\pi + 8\pi = 24\pi$$