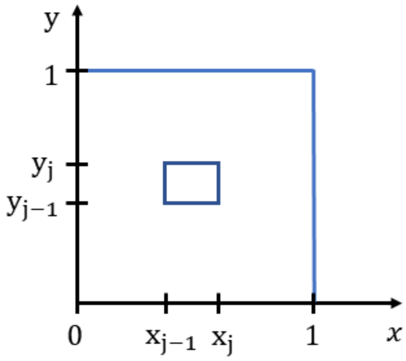
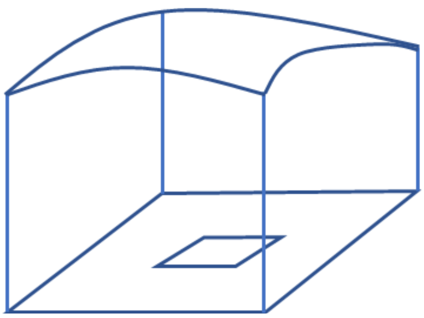
# Module Multiple integrals

## 1. Double integral

문제) 아래의 영역위에 정의된 함수의 그래프 아래의 영역의 부피를 구하여라




$$R = \{(x,y) : 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$$



Find volume under the graph $z = f(x,y)$ on $R$ (위의 그림의 영역. 가운데 기둥이 빠진 실린더의 부피)

Step 1) 영역 R을 직사각형들로 분할

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

$$0 = y_0 < y_1 < \cdots < y_m = 1$$

$R_{ij} = \{(x,y) : x_{i-1} \leqq x \leqq x_i, y_{j-1} \leqq y \leqq y_j\}$ => (i,j)번째 직사각형

Take $P_{ij} = (x_i^*, y_j^*) \in R_{ij}$ :sample point의 선택

$\triangle R_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$ :직사각형의 면적

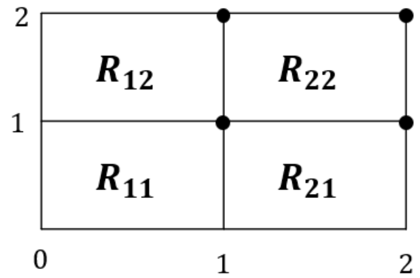
$\sum_{j=1}^{m}\sum_{i=1}^{n} f(P_{ij})\triangle R_{ij}$ 직사각형을 베이스로 하는 사각기둥의 부피

$$\iint_R f dxdy : \ = \lim_{n,m\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(P_{ij})\triangle R_{ij}$$

영역의 부피는 근사 사각기둥의 부피를 전부 다 합한 값과 유사하다. 직사각형을 더 잘게 자르면 각 사각기둥은 실제 기둥과 부피에서 가까워진다 (함수의 연속성의 가정이 필요함). 극한을 취했을 때 그 극한 값을 적분으로 정의하고 표시

예제)

$$f(x,y) = 16 - x^2 - y^2$$
$$R = [0,2]\times[0,2]$$




$$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2$$

$$y_0 = 0 \quad y_1 = 1 \quad y_2 = 2$$

$$R_{11} = [0,1]\times[0,1]$$
$$R_{12} = [x_0, x_1]\times[y_1, y_2]$$
$$= [0,1]\times[1,2]$$
$$R_{21} = [1,2]\times[0,1]$$
$$R_{22} = [1,2]\times[1,2]$$
$$\sum_{j=1}^{2}\sum_{i=1}^{2} f(P_{ij})\triangle R_{ij}$$
$$P_{11} = (1,1), P_{21} = (2,1)$$

$$P_{12} = (1,2), P_{22} = (2,2)$$
$$f(P_{11}) + f(P_{12}) + f(P_{21}) + f(P_{22})$$

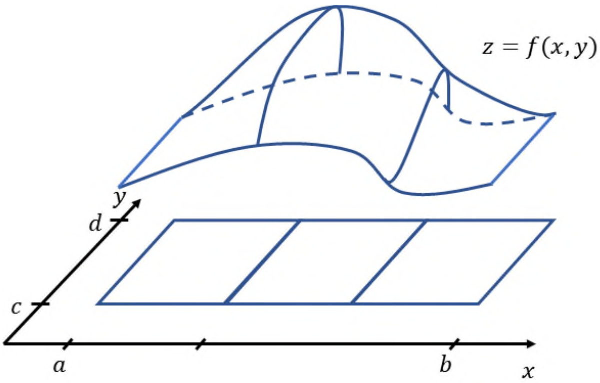
$$= f(1,1) + f(1,2) + f(2,1) + f(2,1)$$

## 2. "Iterated integrals"

실질적으로 적분을 계산하는 법

$f$ : continuous on $[a,b]\times[c,d]$

$\int_c^d f(x,y)dy$ $\quad x$ is fixed




$$A(x) = \int_c^d f(x,y)dy$$

$$\int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx$$

$$= \int_a^b \int_c^d f(x,y)dydx$$

예제)

$$\int_0^3 \int_1^2 x^2 ydydx$$

$$= A(x) = \int_1^2 x^2 ydy = x^2 \int_1^2 ydy$$

$$= x^2 \frac{1}{2} y^2 \Big|_1^2$$

$$= x^2 \frac{1}{2}(4-1) = \frac{3}{2}x^2$$

$$\int_0^3 A(x)dx = \int_0^3 \frac{3}{2}x^2 dx$$

$$= \frac{3}{2}\frac{1}{3}x^3\Big|_0^3 = \frac{1}{2}(27-0) = \frac{27}{2}$$

$$\int_1^2 \int_0^3 x^2 ydxdy = \int_1^2 B(y)dy$$

$$B(y) = \int_0^3 x^2 y dx$$

$$B(y) = y\frac{1}{3}x^3\Big|_0^3 = y\frac{1}{3}(27-0) = 9y$$

$$\int_1^2 9y dy = \frac{9}{2}y^2\Big|_1^2 = \frac{9}{2}(4-1) = \frac{27}{2}$$

Q : $\int_a^b \int_c^d f(xy) dy dx = \int_c^d \int_a^b f(x,y) dx dy$ ?

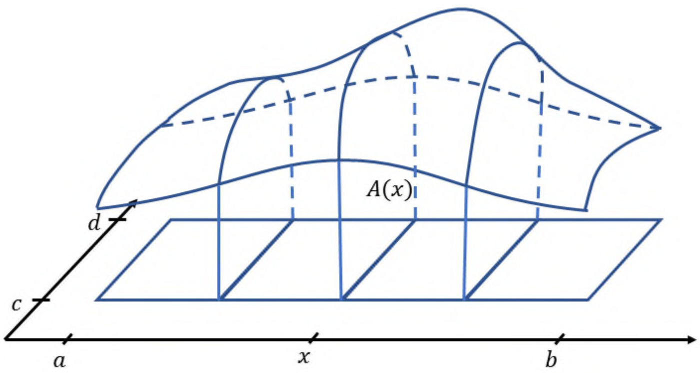
**(Fubini 의 정리)**

위의 질문에 대한 답은 <u>Yes</u> if $f$ is continuous on $[a,b] \times [c,d]$.

Furthermore

$$\int_{[a,b]\times[c,d]} f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$$

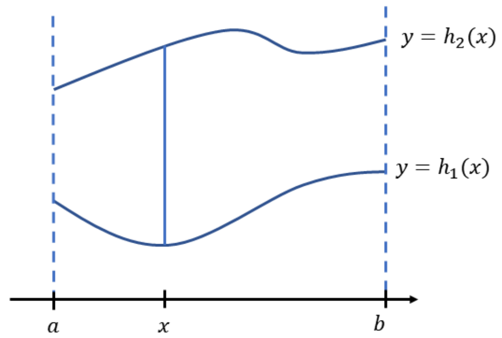
$$= \int_c^d \int_a^b f(x,y) dx dy$$




$$\lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \triangle x = \int_a^b A(x) dx$$

$$= \int_a^b \int_c^d f(x,y) dy dx$$

Rectangle 이 아닌 영역에서의 중적분



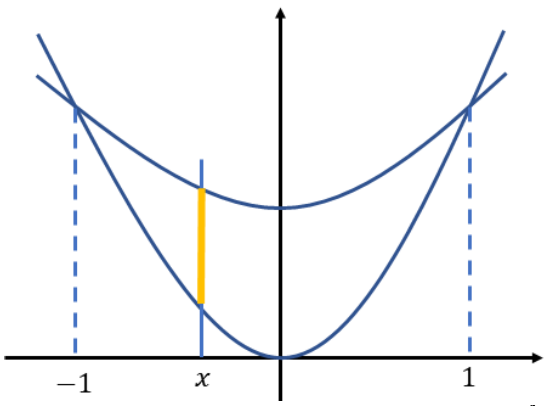

$$R=\{(x,y): a\leqq x\leqq b, h_1(x)\leqq y\leqq h_2(x)\}$$

$$A(x)=\int_{h_1(x)}^{h_2(x)} f(x,y)dy$$

$$\iint_R f dxdy := \int_a^b A(x)dx = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$$

예제)

$$R=\{(x,y): 2x^2\leqq y\leqq 1+x^2\}$$




$$2x^2=1+x^2$$

$$x^2=1$$

$$x=\pm 1$$

$$\iint_R (x+2y)dxdy=$$

$$A(x)=\int_{2x^2}^{1+x^2}(x+2y)dy = xy+y^2\Big|_{y=2x^2}^{y=1+x^2}$$

$$=x(1+x^2)+(1+x^2)^2-[x(2x^2)+(2x^2)^2]$$

$$=x(1-x^2)+(1+2x^2+x^4-4x^4)$$

$$=x-x^3+1+2x^2-3x^4$$

$$\int_{-1}^{1} A(x)dx = \int_{-1}^{1} 1+x+2x^2-x^3-3x^4 dx$$

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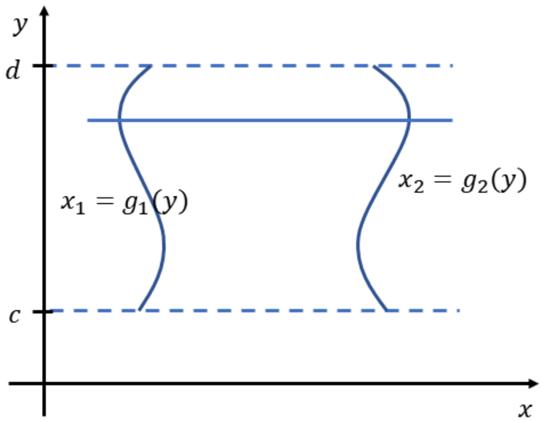
일반적으로 x방향으로 먼저 적분할 것인지 y방향으로 먼저 적분할 것이지를 결정. x방향으로 적분할 경우는 아래와 같이 한다:

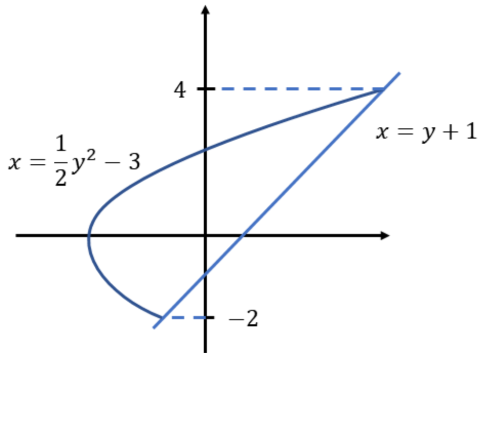



$$R = \{(x,y) : c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$$

$$\iint_R f(x,y)dy := \int_c^d B(y)dy \qquad B(y) = \int_{g_1(y)}^{g_2(y)} f(x,y)dx$$

$$= \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$$

예제) $R$ is bounded by $y = x-1$ and $y^2 = 2x+6$



$$x=\frac{y^2-6}{2}=\frac{1}{2}y^2-3$$

$$y=2(y+1)+6$$

$$y^2-2y-8=(y-4)(y+2)=0$$

$$\int_R xydxdy=?$$

$$B(y)=\int_{\frac{1}{2}y^2-3}^{y+1} f(x,y)dx=\int_{\frac{y^2}{2}-3}^{y+1} xydx$$

$$=y\frac{1}{2}x^2\Big|_{\frac{y^2}{2}-3}^{y+1}$$

$$=\frac{y}{2}\left[(y+1)^2-\left(\frac{y^2}{2}-3\right)^2\right]$$

$$\frac{1}{2}y\left(y^2+2y+1-\left(\frac{y^4}{4}-3y^2+9\right)\right)$$

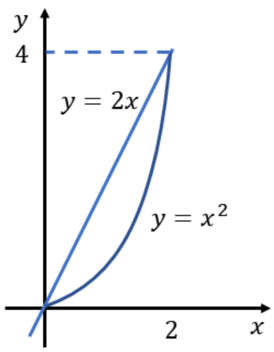
$$\frac{1}{2}y\left(-\frac{y^4}{4}+4y^2+2y-8\right)$$

$$\int_{-2}^{4} B(y)dy=\frac{1}{2}\int_{-2}^{4}-\frac{y^5}{4}+4y^3+2y^2-8ydy$$

예제) $R$ is bounded by the line $y=2x$ and the parabola $y=x^2$

Find $\iint_R x^2+y^2dxdy$

Sol)



$$yx = x^2$$

$$x^2 - 2x = 0$$

$$x = 0,2$$

$$R = \{(x,y) : 0 \leqq x \leqq 2, x^2 \leqq y \leqq 2x\}$$

$$\iint_R x^2 + y^2 dxdy = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dydx$$

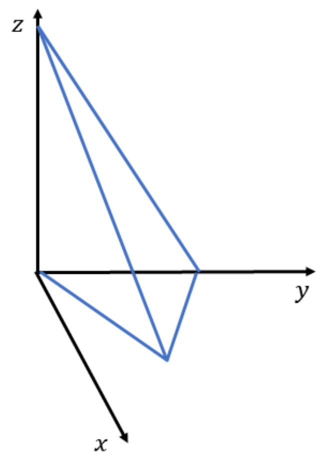
$R$은 다음과 같이 쓰여 질 수 있다.

$$R = \left\{(x,y) : 0 \leqq y \leqq 4, \frac{1}{2}y \leqq x \leqq \sqrt{y}\right\}$$

$$\iint_R x^2 + y^2 dxdy = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2) dxdy$$

Q : 어느 쪽이 더 계산이 간단한가?

연습문제) Volume of the tetrahedron bdd by the planes




$x + 2y + z = 2, x = 2y, x = 0, z = 0$

## 3. Triple integrals

적분 영역이 삼차원 유클리드 공간의 부분집합으로 정의된다.

대표적인 영역은 box이다. 즉 직육면체 내부이다. 아이디어는 직육면체를 작은 박스들로 분할하는 것이다.

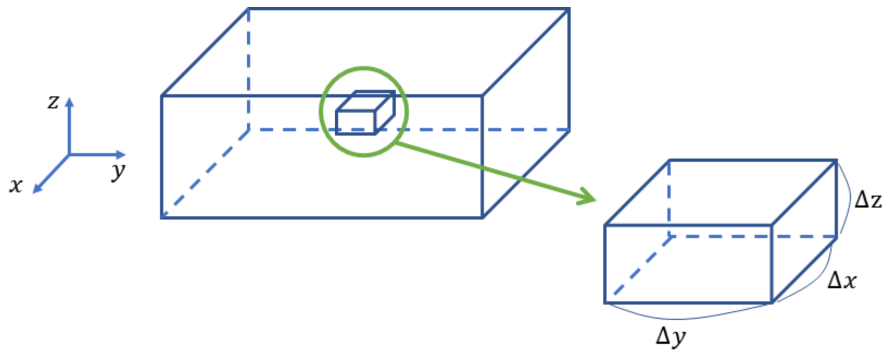
$$D = \{(x,y,z) : a \leq x \leq b, c \leq y \leq s, r \leq z \leq s\}$$

$$a = x_0 < x_1 < \cdots < x_l = b$$

$$c = y_0 < y_1 < \cdots < y_m = d$$

$$r = z_0 < z_1 < \cdots < z_n = s$$

$$D_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$




$$P_{ijk} = (x_i^*, y_j^*, z_k^*)$$

$$\sum_{k=1}^{n} \sum_{j=1}^{m} \sum_{i=1}^{l} f(P_{ijk}) \Delta x \Delta y \Delta z \quad \xrightarrow[l,m,n \to \infty]{} \quad \iiint_D f(x,y,z)\,dxdydz$$

(여기서 극한은 분할 개수를 늘리는 것이다. 각 선분을 분할할 때 분할된 조각의 크기가 점점 작아지도록 분할 한다. 가령 $\Delta x = (b-a)/l$

### Fubini 정리

$f$가 $D = [a,b] \times [c,d] \times [r,s]$ 에서 연속일 때

$$\iiint_D f(x,y,z)\,dxdydz = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dxdydz$$

예제)

$$D=[0,1]\times[-1,2]\times[0,3]$$

$$f=xyz^2$$

$$\iiint_D xyz^2 dxdydz=\int_0^3\int_{-1}^2\int_0^1 xyz^2 dxdydz$$

$$=\int_0^3\int_{-1}^2 \left.\frac{x^2yz^2}{2}\right|_0^1 dydz$$

$$=\int_0^3\int_{-1}^2 \frac{1}{2}y^2 dydz$$

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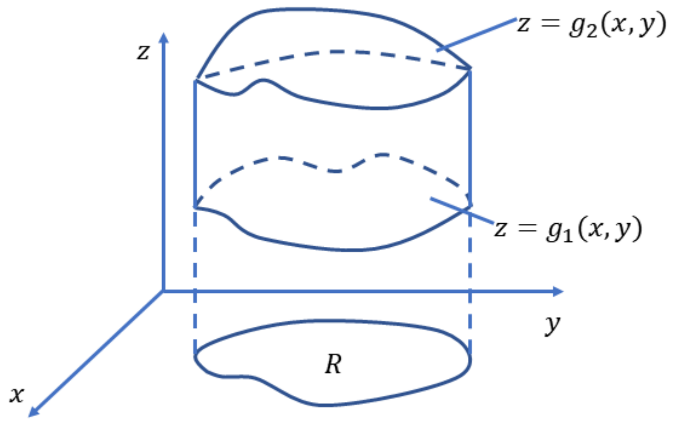
$D$가 직육면체가 아닐 때는 어떻게 할 것인가?

Type 1. $D=\{(x,y,z):(x,y)\in R,\ g_1(x,y)\leq\ z\leq\ g_2(x,y)\}$



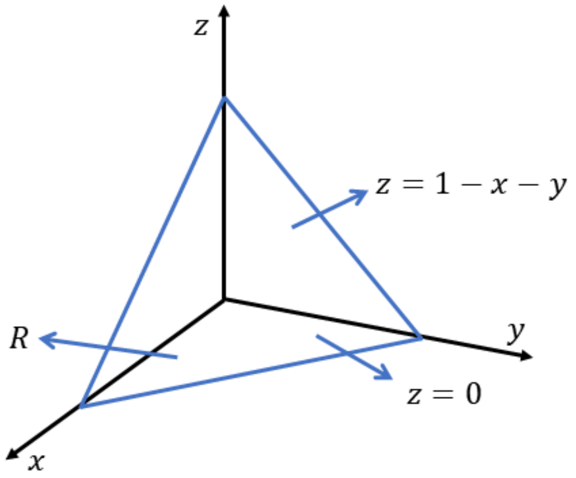

$$\iiint_D f(x,y,z)dV=\iint_R\left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\right]dxdy$$

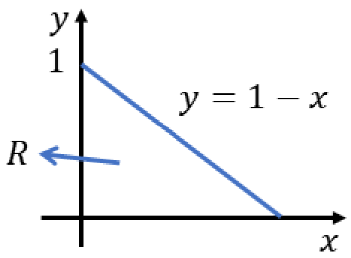
예제) $D$ is bounded by the four planes $x=0,y=0,z=0$ and $x+y+z=1$

$\iiint_D zdV=?$

$$D=\{(x,y,z):(x,y)\in R,\quad 0\leq\ z\leq\ 1-x-y\}$$

$$R=\{(x,y):x\geq\ 0,y\geq\ 0,x+y\leq\ 1\}$$




$$R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$$

$$\iiint_D z\,dV = \iint_R \left[\int_0^{1-x-y} z\,dz\right] dx\,dy$$

$$= \iint_R \frac{1}{2} z^2 \Big|_0^{1-x-y} dx\,dy$$

$$= \iint_R \frac{1}{2}(1-x-y)^2 dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{1}{2}(1-x-y)^2\right] dy\,dx$$

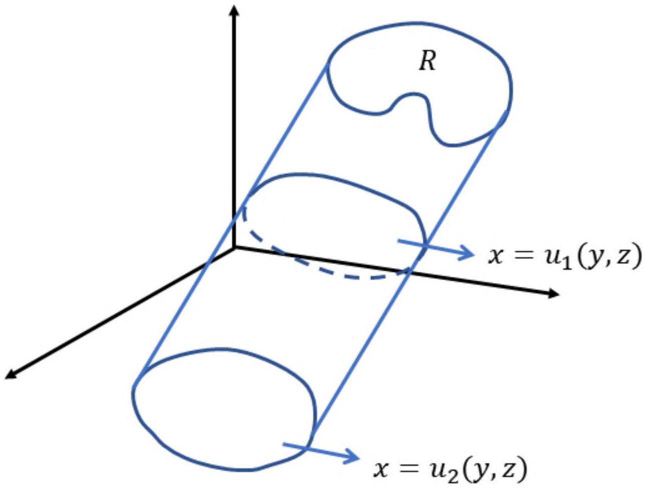
$$= \int_0^1 -\frac{1}{6}(1-x-y)^3 \Big|_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{24}$$

Type 2.

$$D = \{(x,y,z) : (y,z) \in R, u_1(y,z) \leq x \leq u_2(y,z)\}$$

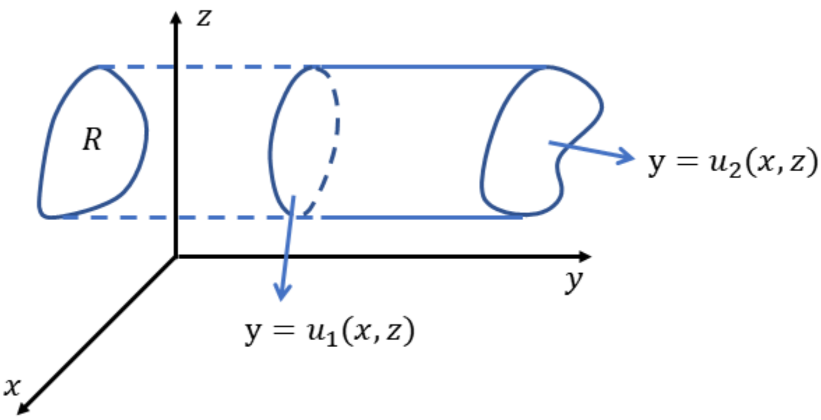
예제)




$$\iiint_D f(x,y,z)dV = \iint_R \left[ \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right] dydz$$

Type 3.

$$D = \{(x,y,z) : (x,y) \in R,\ u_1(x,z) \leqq y \leqq u_2(x,z)\}$$

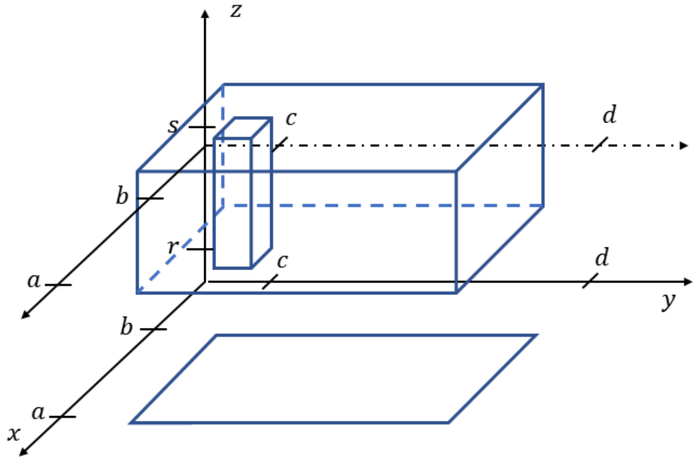



$$\iiint_D f(x,y,z)dV = \iint_R \left[ \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right] dxdz$$

$$(x,y) \mapsto \int_{z=r}^{z=s} f(x,y,z)dz$$

$$\iint_{[a,b]\times[c,d]} \int_{z=r}^{z=s} f(x,y,z)dzdxdy$$

예제)




$$D=[0,1]\times[-1,2]\times[0,3]$$

$$f(x,y,z)=xyz^2$$

$$\iiint_D f(x,y,z)dxdydz$$

$$\int_{z=0}^{z=3} f(x,y,z)dz=\int_{z=0}^{z=3} xyz^2dz$$

$$=xy\frac{1}{3}z^3\Big|_{z=0}^{z=3}=9xy$$

$$\iint_{[0,1]\times[-1,2]}\left[\int_{z=0}^{z=3} f(x,y,z)dz\right]dxdy$$

$$=\iint_{[0,1]\times[-1,2]} 9xydxdy$$

Fubini 정리

$$\iiint_D f(x,y,z)dV=\iint_{[c,d]\times[r,s]}\int_{x=a}^{x=b} f(x,y,t)dxdydz$$

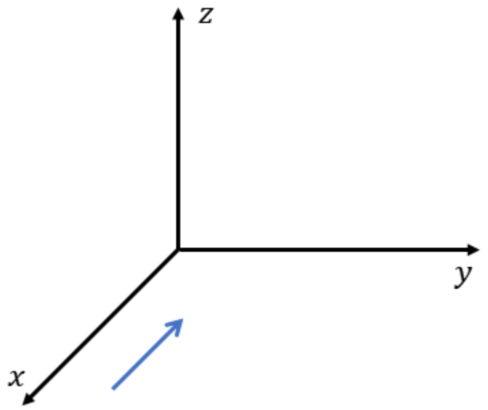
$$=\iint_{[a,b]\times[r,s]}\int_{y=c}^{y=d} f(x,y,z)dydxdz$$

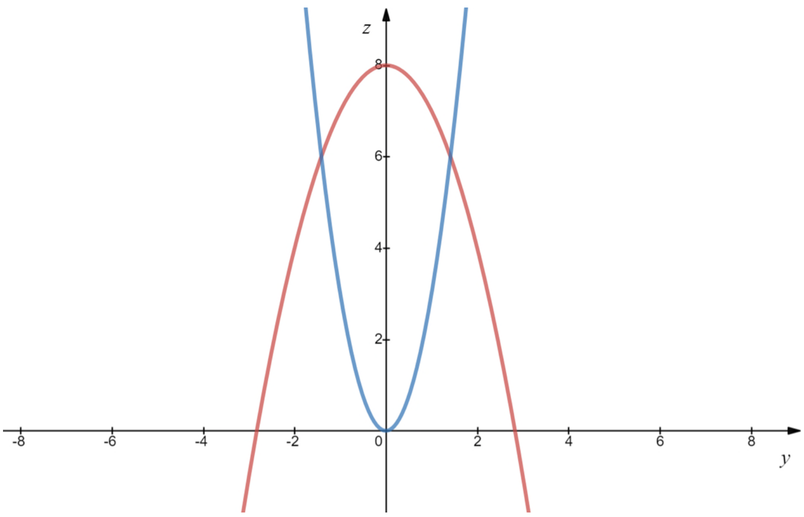
예제) 위로는 $z=8-x^2-y^2$ 아래로는 $z=x^2+3y^2$로 둘러싸인 입체의 부피

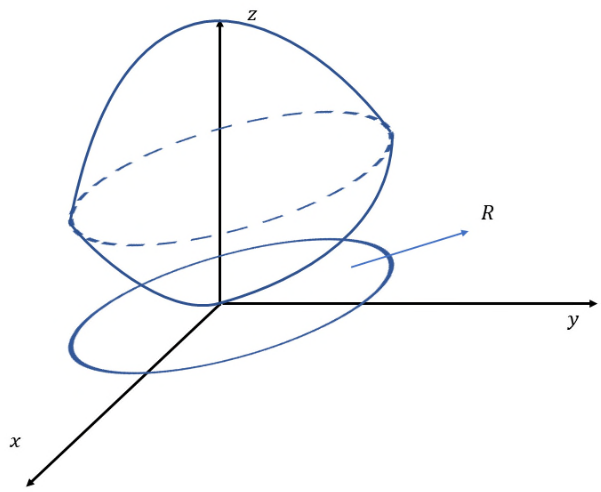
① $D$

② Volume $=\iiint_D 1dV$

$$x=0, z=8-y^2, z=3y^2$$




$$8-x^2-y^2=x^2+3y^2$$

$$2x^2+4y^2=8 \qquad x^2+2y^2=4$$

$$\frac{x^2}{4}+\frac{y^2}{2}=1$$

$$D=\{(x,y,z):(x,y)\in R, x^2+3y^2 \leqq z \leqq 8-x^2-y^2\}$$

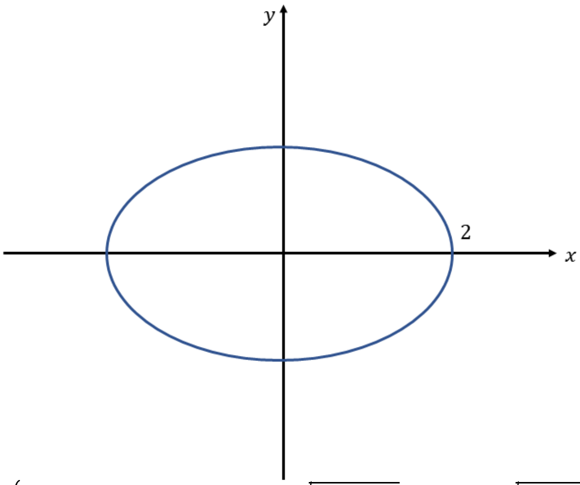
$$\text{volume} = \iiint_D dxdydz$$

$$= \iint_R \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$$

$$\iint_R (8-x^2-y^2)-(x^2+3y^2)dxdy$$

$$\iint_R 8-2x^2-4y^2dxdy$$

$$R=\{(x,y):x^2+2y^2 \leqq 4\}$$




$$R=\left\{(x,y):-2 \leqq x \leqq 2, -\sqrt{\frac{4-x^2}{2}} \leqq y \leqq \sqrt{\frac{4-x^2}{2}}\right\}$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-2x^2-4y^2)dydx$$

$$=\int_{-2}^{2} 8y-2x^2y-\frac{4}{3}y^3\Big|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}dx$$

$$=\int_{-2}^{2}(8-2x^2)(2)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left[\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}+\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}\right]dx$$

$$=\int_{-2}^{2}\sqrt{2}(8-2x^2)\sqrt{4-x^2}-\frac{4}{3}\cdot\ 2\cdot\ \frac{4-x^2}{2}\sqrt{\frac{4-x^2}{2}}\,dx$$

$$=\int_{-2}^{2}\sqrt{2}\sqrt{4-x^2}\left(2(4-x^2)-\frac{2}{3}(4-x^2)\right)dx$$

$$= \int_{-2}^{2} \sqrt{2}\, \frac{4}{3}(4 - x^2)\sqrt{4 - x^2}\, dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{\frac{3}{2}}\, dx$$