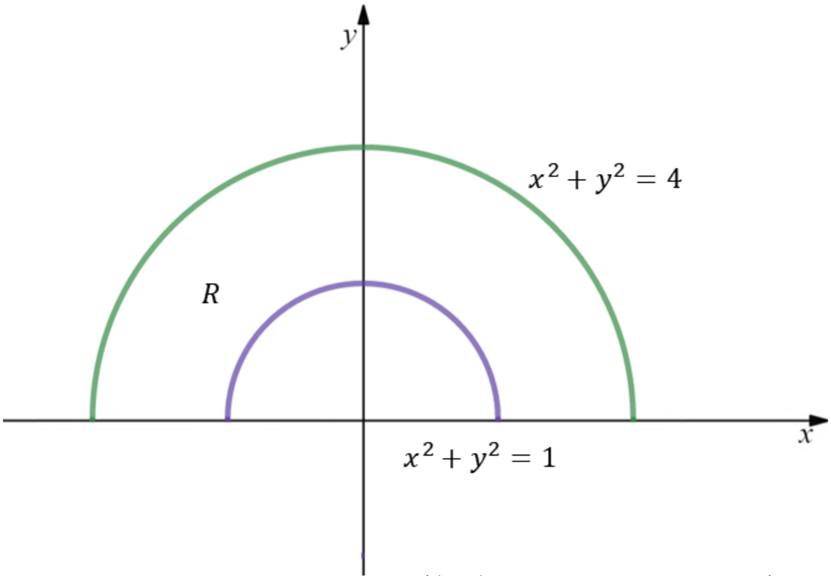
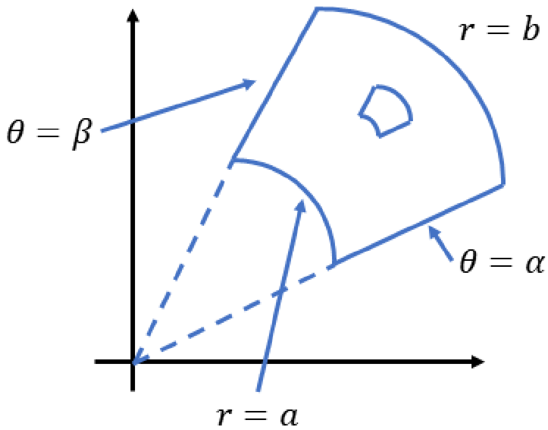
Module Multiple integral in polar and spherical coordinates

1. Double integrals in Polar coordinates



$R=\{(r,\theta):1\leqq r\leqq 2, 0\leqq \theta\leqq \pi\}$

How to evaluate $\iint_R f(x,y)dxdy$?



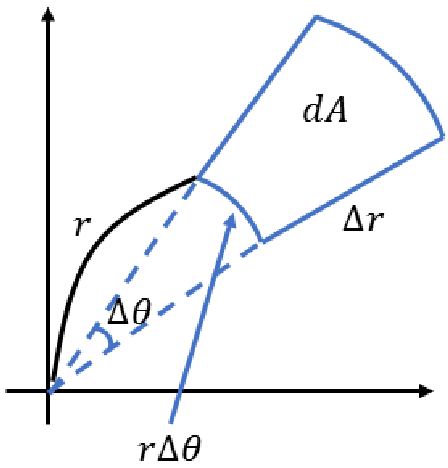
$R=\{(r,\theta):a\leqq r\leqq b, \alpha\leqq \theta\leqq \beta\}$

area $dA=r\triangle\theta\triangle r \rightarrow rdrd\theta$

$r=a=r_0<r_1<\cdots<r_n=b$

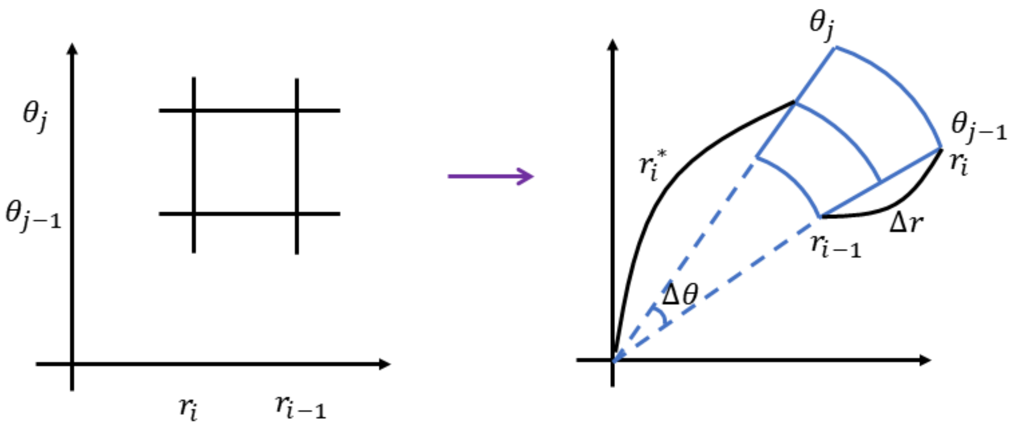
$\alpha=\theta_0<\theta_1<\cdots<\theta_m=\beta$

$(r_i^*,\theta_j^*)\in[r_{i-1},r_i]\times[\theta_{j-1},\theta_j]$



$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i^*,\theta_j^*)\triangle A_{ij}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i^*,\theta_j^*)r_i^*\triangle\theta\triangle r$$



$$\iint_R f(x,y)dxdy = \lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i^*,\theta_j^*)r_i^*\triangle r\triangle\theta \qquad g(r,\theta)=f(r\cos\theta, r\sin\theta)$$

$$=\iint_{[a,b]\times[\alpha,\beta]} g(r,\theta)rdrd\theta$$

$$=\iint_{[a,b]\times[\alpha,\beta]} f(r\cos\theta, r\sin\theta)rdrd\theta$$

예제)

$$R=\{(x,y): 1\leq x^2+y^2\leq 4, y\geq 0\}$$

$$=\{(r,\theta): 1\leq r\leq 2, 0\leq\theta\leq\pi\}$$

$$\iint_R 3x+4ydxdy$$

$$\int_{[1,2]\times[0,\pi]} 3r\cos\theta+4r\sin\theta rdrd\theta$$

$$=\int_0^\pi\int_1^2(3r\cos\theta+4r\sin\theta)rdrd\theta$$

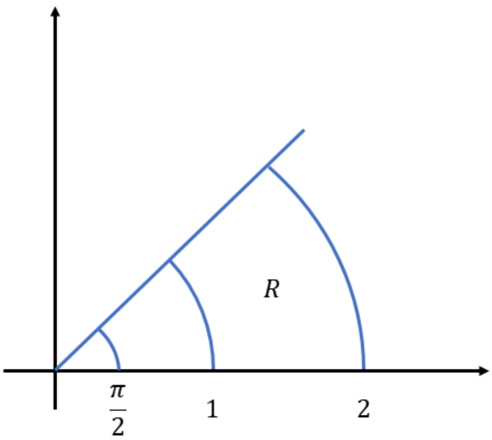
$$=\int_0^\pi\int_1^2 3r^2\cos\theta dr+\int_1^2 4r^2\sin\theta drd\theta$$

$$=\int_0^\pi\left(r^3\Big|_1^2\right)\cos\theta d\theta+\int_0^\pi\left(\frac{4r^3}{3}\Bigg|_0^\pi\right)\sin\theta d\theta$$

$$=7\int_0^\pi\cos\theta d\theta+\frac{28}{3}\int_0^\pi\sin\theta d\theta$$

$$=0+\frac{28}{3}\times2=\frac{56}{3}$$

예제)



$$\iint_R\frac{1}{(x^2+y^2)^{\frac{3}{2}}}dxdy$$

$$\int_0^{\frac{\pi}{4}}\int_1^2 \frac{1}{(r^2)^{\frac{3}{2}}} r dr d\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_1^2 r^{-2} dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_1^2 r^{-2} dr$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{2}\right)$$

$$= \frac{\pi}{8}$$

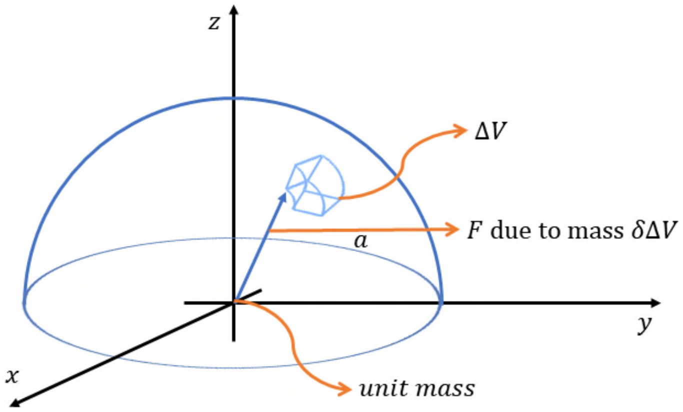
2. Integration in Spherical coordinates

예제) (motivational)

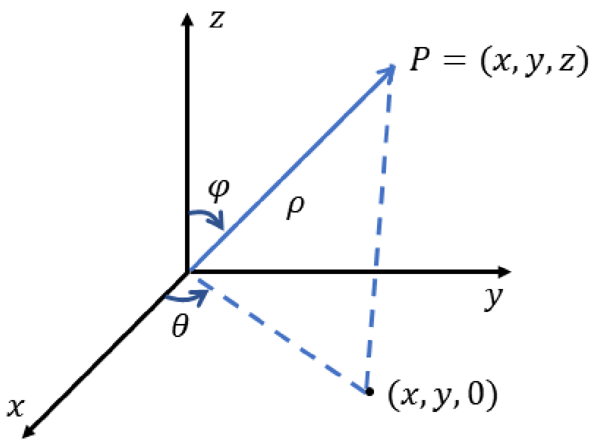
반지름이 a인 Solid hemisphere

density = δ

hemisphere 의 바닥의 중심에 놓인 a unit mass 에 작용하는 중력을 계산하라.



Spherical coordinates



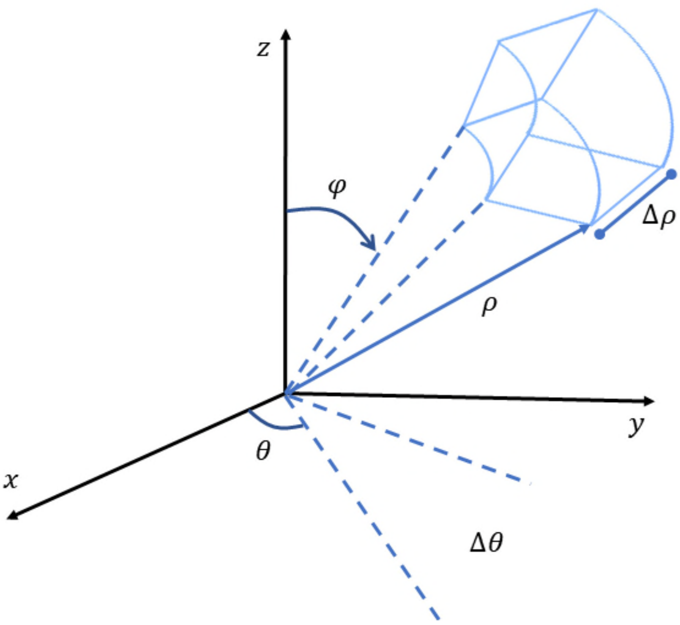
$$x = \rho \sin\phi \cos\theta$$

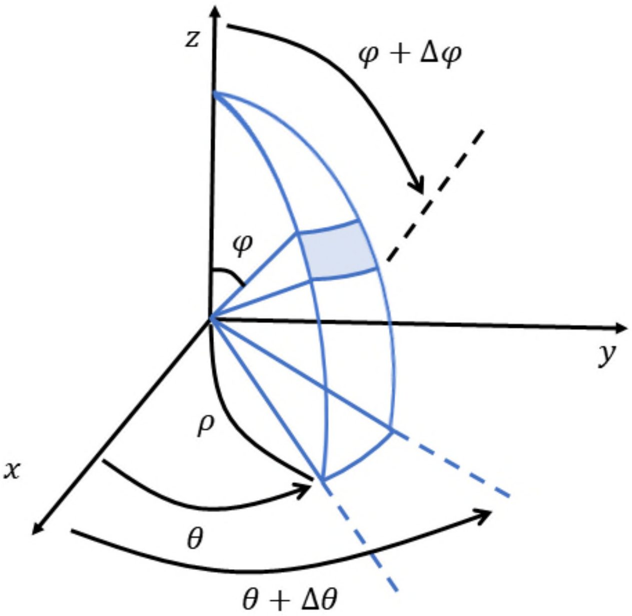
$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$0 \leqq \rho < \infty\ ,\ 0 \leqq \phi \leqq \pi,\ 0 \leqq \theta \leqq 2\pi$$

$$\rho^2 = x^2 + y^2 + z^2$$





$$\text{사각형의 area} = (\rho\triangle\phi)(\rho\triangle\theta\sin\phi)$$

$$\text{volume element} = \triangle\rho(\rho\triangle\phi)(\rho\triangle\theta\sin\phi)$$

$$= \rho^2\sin\phi\triangle\rho\triangle\phi\triangle\theta$$

$$\iiint_D f(x,y,z)dV$$

$$= \iiint_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta$$

Sol of example

F의 x,y 방향은 반구의 대칭성으로 인해 상쇄, 목표한 힘은 z방향 뿐임.

$$\text{Total gravitational force} \approx \sum G\frac{\triangle V\cdot\ (1)}{\rho^2}\cos\phi$$

$$=\iiint\frac{G\cos\phi}{\rho^2}dV$$

$$0\leqq\ \rho\leqq\ a,\ 0\leqq\ \phi\leqq\ \frac{\pi}{2},\ 0\leqq\ \theta\leqq\ 2\pi$$

$$F=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a G\frac{\cos\phi}{\rho^2}\rho^2\sin\phi d\rho d\phi d\theta$$

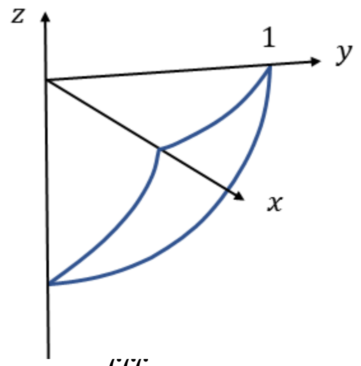
$$=G\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\int_0^a d\rho\right]\frac{1}{2}\sin(2\phi)d\phi d\theta$$

$$=G\int_0^{\frac{\pi}{2}}\frac{a}{2}\sin(2\phi)d\phi\times\int_0^{2\pi}d\theta$$

$$=2\pi G\times\frac{a}{2}\int_0^{\frac{\pi}{2}}\sin(2\phi)d\phi$$

예제) 반지름이 1인 구에서 도려낸 cone $\phi=\frac{\pi}{3}$의 부피

예제)



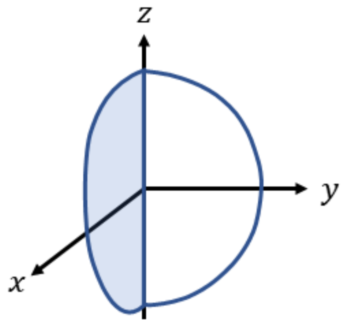
$$\iiint_D\delta(x,y,z)dV$$

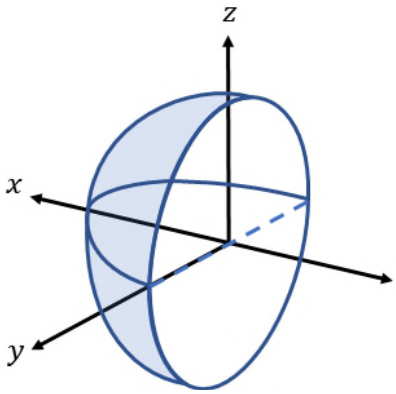
$$D=\left\{0\leqq\ \rho\leqq\ 1,\frac{\pi}{2}\leqq\ \phi\leqq\ \pi,0\leqq\ \theta\leqq\ \frac{\pi}{2}\right\}$$

예제)

$$\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}dydzdx$$

$$D=\left\{0 \leqq \rho \leqq 1, 0 \leqq \phi \leqq \pi, -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right\}$$





예제)

$$0 \leqq \theta \leqq 2\pi,\ 0 \leqq \phi \leqq \frac{\pi}{4},\ 1 \leqq \rho \leqq 2$$

$$D=\iiint_D \sin\phi dV$$

예제)

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^1 f(\rho,\phi,\theta)\rho^2 \sin\phi d\rho d\phi d\theta$$