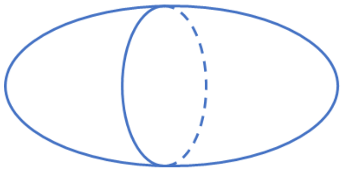
Module Divergence theorem

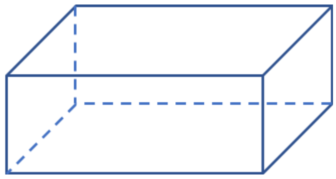
1. Divergence theorem

R : Simple solid region

∂R : Simple closed surface

(예 : ellipsoid, rectangular box)





F는 R을 포함하는 영역에서 정의된 vector field.

n : outward normal vector of R

$$\iint_{\partial R} F \cdot n d\sigma = \iiint_{R} \mathrm{div}\mathrm{F} dV$$

$$F = F_1 i + F_2 j + F_3 k$$

$$\mathrm{divF} = \frac{\partial \mathrm{F}_1}{\partial \mathrm{x}} + \frac{\partial \mathrm{F}_2}{\partial \mathrm{y}} + \frac{\partial \mathrm{F}_3}{\partial \mathrm{z}}$$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\nabla \cdot F = \mathrm{divF}$$

예제) R : $z=0$, $y=0$, $y=2$, $z=1-x^2$ 둘러싸인 영역

$$F = (x + \cos y) i + (y + \sin z) j + (z + e^x) k$$

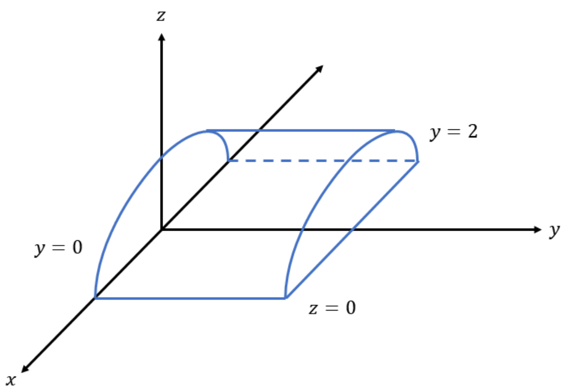
$$\iint_{\partial R} F \cdot n d\sigma = ?$$

$$= \iiint_{R} \mathrm{divF} dV$$

$$\mathrm{divF} = \frac{\partial}{\partial \mathrm{x}}(\mathrm{x} + \cos y) + \frac{\partial}{\partial y}(y + \sin z) + \frac{\partial}{\partial z}(z + e^x)$$

$$= 1 + 1 + 1 = 3$$

$$R = \{(x, y, z) : -1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1 - x^2\}$$



$$\iiint_R 3dV = \int_0^2 \int_{-1}^1 \int_0^{1-x^2} 3dzdxdy$$

$$= \int_0^2 \int_{-1}^1 3z \Big|_0^{1-x^2} dxdy$$

$$= \int_0^2 \int_{-1}^1 3(1-x^2)dxdy$$

$$= \int_0^2 3\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]dy$$

$$= 2\cdot\ 3\cdot\ 2\left(1-\frac{1}{3}\right) = 8$$

Divergence 정리의 example

예제)

$$\mathrm{div}\vec{\mathrm{F}} = 0$$

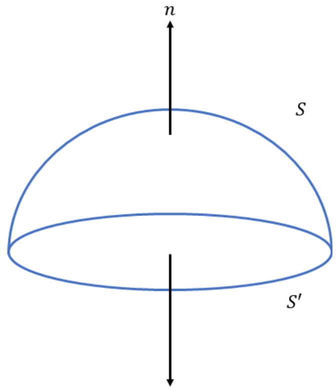
$$\vec{F} = x\vec{i} - 2y\vec{j} + (z+1)\vec{k}$$

$$\mathrm{div}\vec{\mathrm{F}} = 1-2+1 = 0$$

$$S: x^2+y^2+z^2 = 1 \qquad z \geqq 0$$

$$\iint_S \vec{F}\cdot\ \vec{n}d\sigma$$

$$\int_{S\cup S'} \vec{F}\cdot\ \vec{n}d\sigma = \int_R \mathrm{div}\vec{\mathrm{F}}dV = 0$$



$$\int_S \vec{F}\cdot \vec{n}d\sigma = -\int_{S'} \vec{F}\cdot \vec{n}d\sigma$$

$$= \int_{S'} \vec{F}\cdot (-\vec{n})d\sigma$$

$$\int_{\{z=0,x^2+y^2\leqq 1\}} \vec{F}\cdot \vec{k}d\sigma$$

$$= \iint_{\{x^2+y^2\leqq 1\}} (z+1)dxdy = \iint_D dxdy = \pi$$

예제)

$$\iint_S x^2+y+zd\sigma$$

$$\iiint_R \mathrm{div}\vec{\mathrm{F}}dV = \iint_{\partial R} \vec{F}\cdot \vec{n}d\sigma$$

$$F_1x+F_2y+F_3z = x^2+y+z$$

$$F_1 = x,\ F_2 = 1,\ F_3 = 1$$

$$\mathrm{divF} = 1$$

$$\iiint_R dV = \iint_{\partial R} x^2+y+zd\sigma = \frac{4}{3}\pi$$

예제)

$$\vec{F} = 2x\vec{i}+y^2\vec{j}+z^2\vec{k}$$

$$\iint_S \vec{F}\cdot \vec{n}d\sigma \qquad S: x^2+y^2+z^2 = 1$$

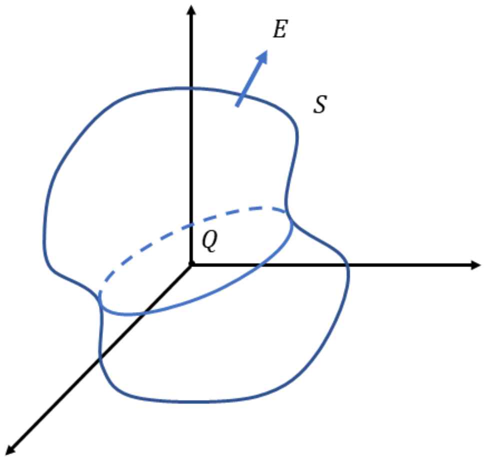
$$\iiint_R \mathrm{div}\vec{F}dV = \iiint_R (2+2y+2z)dV$$

$$= \iiint_R 2dV + 2\iiint_R ydV + 2\iiint_R zdV$$

$$= 2vol(R)$$

$$\iint_{\{x^2+z^2 \leq 1\}} \int_{y=-\sqrt{1-x^2-y^2}}^{y=\sqrt{1-x^2-y^2}} ydydxdz = 0$$

2. Gauss 의 법칙



S : closed surface

$$\iint_S E \cdot \ n d\sigma = 4\pi k$$

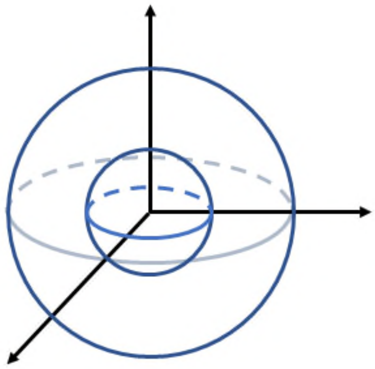
$$E = kq\frac{\vec{r}}{|\vec{r}|^3}$$

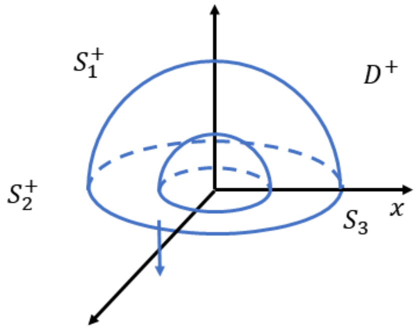
영역이 simple solid region의 union일 때

$$\begin{cases} D = B_2 - \overline{B_1} \\ \partial D = S_1 \cup \ S_2 \end{cases}$$

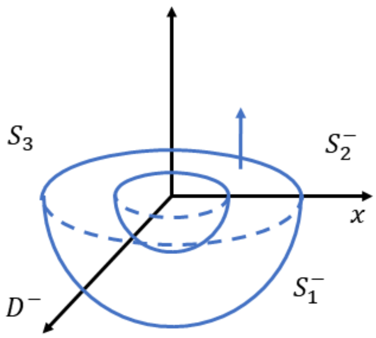
$$\Rightarrow D = D_+ \cup \ D_-$$

$$\partial D_+ = S_1^+ \cup \ S_2^+ \cup \ S_3$$





$$\iiint_{D_+} \mathrm{divF}\, dV = \iint_{S_1^+} F\cdot\ n_1 d\sigma + \iint_{S_2^+} F\cdot\ n_2 d\sigma + \iint_{S_3} F\cdot\ n d\sigma$$



$$\iiint_{D_-} \mathrm{divF}\, dV = \iint_{S_1^-} F\cdot\ n_1 d\sigma + \iint_{S_2^-} F\cdot\ n_2 d\sigma + \iint_{S_3} F\cdot\ (-n) d\sigma$$

$$\iiint_{D} \mathrm{divF}\, dV = \iiint_{D_+} \mathrm{divF}\, dV + \iiint_{D_-} \mathrm{divF}\, dV$$

$$= \iint_{S_1^+ \cup\ S_1^-} F\cdot\ n_1 d\sigma + \iint_{S_2^+ \cup\ S_2^-} F\cdot\ n_2 d\sigma$$

$$= \iint_{S} F\cdot\ n d\sigma$$

예제)

$$D: a^2 \leqq\ x^2 + y^2 + z^2 \leqq\ b^2$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$F(x,y,z) = \frac{xi+yj+zk}{\rho^3}$$

$$\iint_{\partial D} F \cdot \ n d\sigma = \iiint_D F \mathrm{div\,F} dV$$

$$\mathrm{div\,F} = \frac{\partial}{\partial \mathrm{x}}\frac{\mathrm{x}}{\rho^3} + \frac{\partial}{\partial \mathrm{y}}\frac{\mathrm{y}}{\rho^3} + \frac{\partial}{\partial \mathrm{z}}\frac{\mathrm{z}}{\rho^3}$$

$$\frac{\partial}{\partial x}\frac{x}{\rho^3} = \rho^{-3} + (-3)x\rho^{-4}\frac{\partial \rho}{\partial x}$$

$$\rho^2 = x^2+y^2+z^2$$

$$2\rho\frac{\partial \rho}{\partial x} = 2x \qquad \frac{\partial \rho}{\partial x} = \frac{x}{\rho}$$

$$\frac{\partial}{\partial x}\frac{x}{\rho^3} = \rho^{-3} - 3x\rho^{-4}\frac{x}{\rho}$$

$$= \rho^{-3} - 3x^2\rho^{-5}$$

$$\mathrm{div\,F} = \rho^{-3} - 3\mathrm{x}^2\rho^{-5} + \rho^{-3} - 3\mathrm{y}^2\rho^{-5} + \rho^{-3} - 3\mathrm{z}^2\rho^{-5}$$

$$= 3\rho^{-3} - 3\rho^{-5}(x^2+y^2+z^2) = 3\rho^{-3} - 3\rho^{-5} \cdot \ \rho^2 = 0$$

$$\iint_{\partial D} F \cdot \ n d\sigma = 0$$

$$\iint_{S_1} F \cdot \ n_1 d\sigma + \iint_{S_2} F \cdot \ n_2 d\sigma = 0$$

$$\iint_{S_1} F \cdot \ n_1 d\sigma = \iint_{S_2} F \cdot \ (-n_2) d\sigma$$

$-n_2 \rightarrow$ outward on $\sqrt{2}$

3. 전자기학에서의 Gauss 의 law

$$E = \frac{q}{|r|^2}\frac{r}{|r|} = q\frac{r}{|r|^3} = q\frac{xi+yj+zk}{\rho^3}$$

$$\Rightarrow \iint_S E\cdot \ nd\sigma = \iint_{S(\varepsilon)} E\cdot \ nd\sigma$$

$$n = \frac{r}{|r|} = \frac{xi+yj+zk}{\rho}$$

$$E\cdot \ n = q\frac{r}{\rho^3}\cdot \ \frac{r}{\rho} = q\frac{\rho^2}{\rho^4} = \frac{q}{\rho^2}$$

$$\iint_{S(\varepsilon)} \frac{q}{\rho^2} d\sigma = \frac{q}{\varepsilon^2}\iint_{S(\varepsilon)} d\sigma$$

$$= \frac{q}{\varepsilon^2} 4\pi\varepsilon^2 = 4\pi q$$