## Module Taylor approximation

$$(x,y)=(x-a+a,\, y-b+b)=(a,b)+(x-a,\, y-b)$$

$$h=x-a,\, k=y-b$$

$$(x,y)=(a,b)+(h,k)$$

$$\phi(t)=f((a,b)+t(h,k))$$

$$\phi(t)=\phi(0)+\phi'(0)t+\frac{\phi''(c)}{2}t^2$$

$$\phi'=\frac{\partial f}{\partial x}(P+tv)h+\frac{\partial f}{\partial y}(P+tv)k$$

$$\phi'(0)=\frac{\partial f}{\partial x}(P)h+\frac{\partial f}{\partial y}(P)k$$

$$=f_x(P)(x-a)+f_y(P)(y-b)$$

$$\phi''(t)=\frac{\partial^2 f}{\partial x^2}(P+tv)h^2+\frac{\partial^2 f}{\partial y\partial x}(P+tv)hk+\frac{\partial^2 f}{\partial x\partial y}(P+tv)hk+\frac{\partial^2 f}{\partial y^2}(P)k^2$$

$$\phi''(c)=f_{xx}h^2+2f_{xy}hk+f_{yy}k^2\Big|_{P+cv} \qquad 0<c<1$$

$$|\varepsilon| \leqq \ ?$$

$$M=\max_{R}\{|f_{xx}|,|f_{xy}|,|f_{yy}|\}$$

$$|\phi''(c)| \leqq M(h+k)^2$$

$$\leqq 2M(h^2+k^2)=2M(|x-a|^2+|y-b|^2)$$

$$\phi(1)=f(x,y)=f(P)+f_x(P)(x-a)+f_y(P)(y-b)+\frac{1}{2}\phi''(c)$$

$$|\varepsilon(x,y)| \leqq \frac{1}{2}2M(|x-a|^2+|y-b|^2)$$

예제)

$$f(x,y)=\frac{1}{1+x-y} \qquad \text{near } (0,0)$$

$$L(x,y)=f(0,0)+f_x(0,0)x+f_y(0,0)y$$

$$f_x=\frac{-1}{(1+x+y)^2} \qquad f_y=\frac{-1}{(1+x-y)^2}$$

$$L=1-x+y$$

예제)

$$f(x,y) = e^x \sin y$$

$$L = f(0,0) + f_x(0,0)x + f_y(0,0) \qquad |x| < 0,1$$

$$|y| < 0,1$$

$$\varepsilon \leqq M(x^2 + y^2)$$

$$f_x = e^x \sin y,\ f_y = e^x \cos y$$

$$f_{xx} = e^x \sin y,\ f_{xy} = e^x \cos y\ ,\ f_{yy} = -e^x \sin y$$

$$\max|f_{xx}| = \sin y e^x = e^{0.1} < 3^{\frac{1}{2}} < 2$$

$|x| \leqq 0,1 \quad |x| \leqq 0,1$

$|y| \leqq 0,1$

$$M \leqq 2$$

$$\varepsilon \leqq 2(10^{-2} + 10^{-2}) = 4 \times 10^{-2}$$

$$V = \pi r^2 h$$

$$\triangle V = V(r + \triangle r, h + \triangle h) - V(r,h)$$

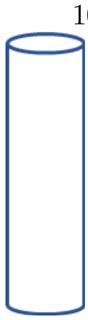
$$\approx V_r \triangle r + V_h \triangle h$$

$$f(a+h, b+k) = f(a,b) + f_x(a,b)h + f_y(a,b)k$$

$$V_r = 2\pi r h \qquad V_h = \pi r^2$$

$$\triangle V \approx 2\pi r h \triangle r + \pi r^2 \triangle h$$

$r = 1 \ \ h = 5 \qquad 10\pi \triangle r + \pi \triangle h$



h fixed, r changes $\Rightarrow 10\pi$

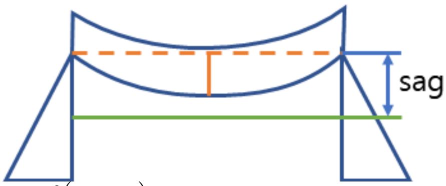
r fixed, h changes $\Rightarrow \pi$

$r = 5 \ \ h = 1 \qquad 10\pi \triangle r + 25\pi \triangle h$

$2\pi r h > \pi r^2 \qquad 2h > r \ \Rightarrow \triangle r$ is more sensitive then $\triangle h$

$2h < r \ \Rightarrow \triangle h$ is more sensitive then $\triangle r$

예제)




$$f(x,y,z)$$

$$f(a+h,b+k,c+l)$$

$$\frac{d}{dt}f(P+tv)=f_xh+f_yk+f_zl$$

$$\left(\frac{d}{dt}\right)^2 f(P+tv)=(f_{xx}h+f_{xy}k+f_{xz}l)h+(f_{yx}h+f_{yy}k+f_{yz}l)k+(f_{zx}h+f_{zy}k+f_{zz}l)l$$

$$=f_{xx}h^2+2f_{xy}hk+2f_{yz}kl+f_{yy}k^2+f_{zz}l^2+2f_{zx}lh$$

$$\sum_{j,k=1}^{3} f_{jk}h_jh_k$$

$$|\varepsilon| \leqq M(h+k+l)^2$$

예제)

$$f(x,y,z)=x^2-xy+3\sin z$$

$$f(2,1,0)=4-2\cdot 1+0=2$$

$$f_x|_P=2x-y|_P=2(2)-(1)=4-1=3$$

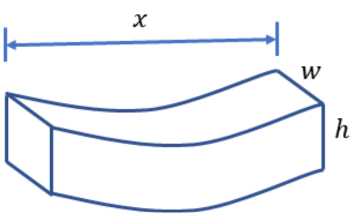
$$f_y|_P=-x|_P=-2$$

$$f_z|_P=3\cos(0)=3$$

$$L=2+3(x-2)+(-2)(y-1)+3z$$

Deflection of loaded beams

$$S(sag)=C\frac{px^4}{wh^3}$$




$$P= \text{ the load } (N/m)$$

$$dS=S_P\,dp+S_x\,dx+S_w\,dw+S_h\,dh$$

$$\text{when }\ x = 4m,\ w = 10\text{cm},\ h = 20\text{cm}$$

$$p = 100N/m$$

$$dS = C\left(\frac{x^4}{wh^3}dp + 4\frac{px^3}{wh^3}dx - \frac{px^4}{w^2h^3}dw - 3\frac{px^4}{wh^4}dh\right)$$

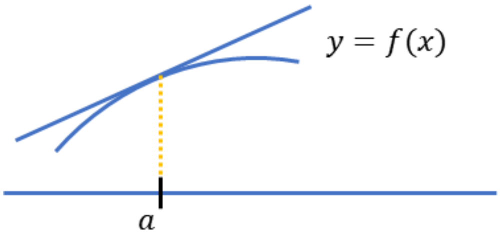
$$1^{st} = C\frac{px^4}{wh^3}\left(\frac{1}{p}dp + \frac{4}{x}dx - \frac{1}{w}dw - \frac{3h}{h}dh\right)$$

$$= S_0\left(\frac{1}{100}dp + \frac{4}{4}dx - \frac{1}{0.1}dw - \frac{3}{0.2}dh\right)$$

1cm wider 보다 1cm higher sag를 더 줄일 수 있다.

일변수 함수의 경우 테일러 근사

$f(x)$를 $x=a$ 근처에서 근사



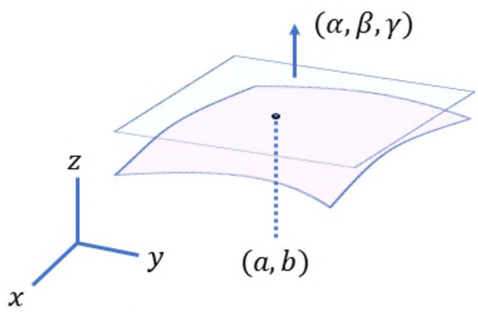

$$y=f(a)+f'(a)(x-a)$$

$f(x)\approx\ a+f'(a)(x-a)\quad \Rightarrow f(x)$를 $x=a$에서 일차식으로 근사.

2 변수 함수의 경우

$f(x,y)$를 $(x,y)=(a,b)$에서 근사

기하학적 접근: 곡면 $z=f(x,y)$을 점$(a,b,f(a,b))$에서 접평면으로 근사




$\alpha(x-a)+\beta(y-b)+\gamma(z-f(a,b))=0$ => $(a,b,f(a,b))$를 지나는 평면의 방정식

$$z-f(a,b)=-\frac{\alpha}{\gamma}(x-a)-\frac{\beta}{\gamma}(y-b)$$

$$z=f(a,b)-\frac{\alpha}{\gamma}(x-a)-\frac{\beta}{\gamma}(y-b)\quad \text{일차함수}$$

$f(x,y)\approx\ f(a,b)+A(x-a)+B(y-b)$ 주어진 함수를 (a,b)에서 근사하는 일차함수 찾기

=> good candidate이 만족해야할 조건 (기준점에서 미분값 일치)

$$\frac{\partial f}{\partial x}(a,b)=A$$

$$\frac{\partial f}{\partial y}(a,b)=B$$

$$\Rightarrow f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

$P=(a,b)$에서의 2차 근사 다항식은 다음과 같다.

$$f(P)+f_x(P)(x-a)+f_y(P)(y-b)+\frac{1}{2!}(f_{xx}(P)(x-a)^2+2f_{xy}(x-a)(y-b)+f_{yy}(P)(y-b)^2)$$

편의상 다음 미분연산자를 도입

$$D=(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}$$

$$D=h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}$$

$$Df(x,y)=hf_x+kf_y$$

$$D^2f(x,y)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)$$

$$=D(hf_x+kf_y)$$

$$=hDf_x+kDf_y$$

$$=h(hf_{xx}+kf_{xy}+k(hf_{yx}+kf_{yy})$$

$$=h^2f_{xx}+hkf_{xy}+khf_{yx}+k^2f_{yy}$$

$$=h^2f_{xx}+2hkf_{xy}+k^2f_{yy}$$

$$D^2=h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}$$

$$D^3=h^3\frac{\partial^2}{\partial x^3}+3h^2k\frac{\partial^2}{\partial x^2\partial y}+3hk^2\frac{\partial^2}{\partial x\partial y^2}+k^3\frac{\partial^3}{\partial y^3}$$

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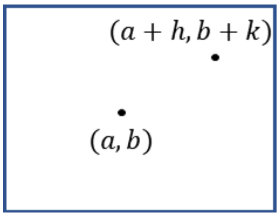
**Taylor 정리 (n차 테일러 근사다항식과 본래 함수와의 차이에 대한 정리)**

$f\in C^n$ ( n번 까지 미분가능하며 도함수들이 모두 연속) => 다음을 만족하는 $c\in(0,1)$가 존재한다:

$$f(a+h,b+k)=f(a,b)+Df(a,b)+\cdots+\frac{D^{n-1}f(a,b)}{(n-1)!}+\frac{D^nf(a+ch,a+ck)}{n!}$$




(proof)

일변수 함수 $\phi(t)=f(P+tv)$에 대해 테일러정리를 바로 적용하면 됨

여기서 $v=(h,k)$

$$\phi(t)=\phi(0)+\phi'(0)t+\frac{\phi''(0)}{2!}t^2+\cdots+\frac{\phi^{n-1}(0)}{(n-1)!}t^{n-1}+\frac{\phi^n(c)}{n!}t^n$$

$$\phi(0)=f(P)$$

$$\frac{d}{dt}f(a+th,b+tk)=f_x(P+tv)h+f_y(P+tv)k$$

$$\left(\frac{d}{dt}\right)^2 f(P+tv)=\frac{d}{dt}f_x(P+tv)h+f_y(P+tv)k$$

$$=(f_{xx}h+f_{xy}k)h+(f_{yx}h+f_{yy}k)k$$

$$=D^2f(P+tv)$$

$$f(P+tv)=f(P)+(f_xPh+f_yPk)t+\cdots$$

위 식에다 $t=1$을 대입하면

$$f(P+V)=f(P)+Df(P)+\frac{D^2f}{2!}(P)$$

$$=\sum_{k=0}^{n-1}\frac{D^kf(P)}{k!}+\frac{\phi^n(c)}{n!}$$

이해를 위해 n=2인 경우를 살펴보면

$$f(a+h,b+k)-f(a,b)$$

$$=f_x(P)h+f_y(P)k+\frac{1}{2}(h^2f_{xx}+2hkf_{xy}+k^2f_{yy})^2(a+ch,b+ck)$$

응용) 함수의 최대 최소에 대한 판별식 정리의 증명

판별식 정리: $Hf|_P > 0 \quad f_x(P)=0,\ f_y(P)=0$

$f_{xx} > 0$ => local min at $P$

$f_{xx} < 0$ => local max at $P$

판별식 정리가 작동하는 이유를 테일러정리로 설명할 수 있다

$$Q(c) = (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})\Big|_{\substack{a+ch \\ b+ck}}$$

$Q(c)$의 sign 결정하기

$$Q(0) = h^2 f_{xx}(P) + 2hk f_{xy}(P) + k^2 f_{yy}(P)$$

$$f_{xx}Q(0) = h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{yy} f_{xx}$$

$$= (hf_{xx})^2 + 2(hf_{xx})(kf_{xy}) + (kf_{xy})^2 - (kf_{xy})^2 + k^2 f_{yy} f_{xx}$$

$$= (hf_{xx} + kf_{xy})^2 + k^2(f_{xx}f_{xy} - f_{xy}^2)$$

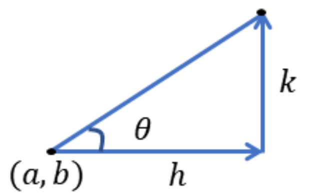
여기서 마지막 항에 f의 헤세 행렬식 $Hf$가 등장한다. 만약 $Hf > 0$이면 $f_{xx}Q(0) > 0$

$$f_{xx} > 0 \Rightarrow Q(0) > 0 \Rightarrow Q(c) > 0 \Rightarrow f(a+h, b+k) - f(a,b) > 0$$

case
1. $Hf > 0,\quad f_{xx} > 0$
2. $Hf > 0,\quad f_{xx} < 0$
3. $Hf < 0,\quad h,k$에 따라서 $Q(0)$의 sign이 달라진다.

$$h^2\left(f_{xx} + \frac{k}{h} f_{xy}\right)^2 + k^2 Hf$$

$$= h^2\left[\left(f_{xx} + \frac{k}{h} f_{xy}\right)^2 + \left(\frac{k}{h}\right)^2 Hf\right]$$




$\dfrac{k}{h} = \tan\theta$ 방향고정. $h,k$를 줄여도 $Q$의 sign은 변하지 않는다.

$$(f_{xx}+\tan\theta f_{xy})^2+(\tan\theta)^2 H_f$$

$$=\tan^2\theta[(\cos\theta f_{xx}+f_{xy})^2+H_f]$$

$$\to \infty \qquad \text{as } \theta\to 0$$

$$\to 0 \qquad \text{as } \theta\to \frac{\pi}{2}$$

4.

$$Hf=0$$

$$f_{xx}Q(0)=(hf_{xx}+kf_{xy})^2$$

$$(f_{xx},f_{xy})\cdot \quad (h,k)$$

$$\text{case} \quad Q(0)=0$$

$Q(c)$의 sign에 대해서 알 수가 없다

예제) $e^x\cos y$의 (0,0)에서의 테일러 다항식을 구하여라.

테일러 정리에 따르면

$$f(x,y)\approx f(a,b)+f_x(P)(x-a)+f_y(P)(y-b)+\frac{1}{2}(f_{xx}(P)(x-a)^2+2f_{xy}(P)(x-a)(y-b)+f_{yy}(P)(y-b)^2)$$

테일러 정리를 바로 적용하기 보다는 알고 있는 것을 활용 즉

$$e^x=1+x+\frac{1}{2!}x^2+\cdots$$

$$\cos y=1-\frac{y^2}{2!}+\frac{y^4}{4!}+\cdots$$

를 이용하면

$$e^x\cos y=(1+x+\frac{x^2}{2!}+\cdots)(1-\frac{y^2}{2!}+\frac{y^4}{4!}+\cdots)$$

$$=1+x+\frac{1}{2}x^2-\frac{1}{2}y^2+(3\text{차식})$$

$$f(x,y)=1+x+\frac{1}{2}x^2+xy-\frac{1}{2}y^2+h\cdot \quad t$$

예 8.7) 유사한 문제를 테일러정리를 적용해서도 해보자.

$$f(x,y)=e^x\sin y$$

$$f_x = e^x \sin y \qquad \sin 0 = 0$$

$$f_y = e^x \cos y \qquad \cos 0 = 1$$

$$(0,0)$$

$$f_{xx} = e^x \sin y \quad \rightarrow \quad \sin 0 = 0$$

$$f_{xy} = e^x \cos y \qquad \cos 0 = 1$$

$$f_{yy} = -e^x \sin y \qquad -\sin 0 = 0$$

$$f(x,y) = P_2(x,y) + E_2(x,y)$$

$$P_2 = f(0,0) + f_x x + f_y y + \frac{1}{2}(f_{xx}x^2 + 2f_{xy} + f_{yy})$$

$$= 0 + 0 + y + \frac{1}{2}(2xy) = y + xy$$

추가 응용으로 $e^{0.01}\sin 0.02$ 의 근사값을 계산해보자. 이 때 $10^{-5}$의 오차의 범위에서 해보자.

$$f(0.01, 0.02) \approx 0.02 + (0.01)(0.02)$$

$$E_2 = \frac{1}{6}(ax^3 + bx^2y + 3cxy^2 + dy^3)$$

$$(a,b,c,d) \sim \partial^{(3)} f$$

$$\Rightarrow \quad |a| \leqq e^{0.01} < 3^{\frac{1}{2}} < 2$$

$$|b|, |c|, |d| \leqq 2$$

$$|E_2| \leqq \frac{1}{6}(|a|(0.01)^3 + 3|b|(0.01)^2(0.02) + 3|c|(0.01)(0.02)^2 + |d|(0.02)^3)$$

$$= \frac{2}{6}(10^{-6} + 3 \cdot 10^{-4} \times 2 \times 10^{-2} + 3 \times 10^{-2} \times 4 \times 10^{-4} + 8 \times 10^{-6})$$

$$= \frac{1}{3}(1 + 6 + 12 + 8) \times 10^{-6}$$

$$= 9 \times 10^{-6} < 10^{-5}$$