Module Substitution rule for Multiple Integral

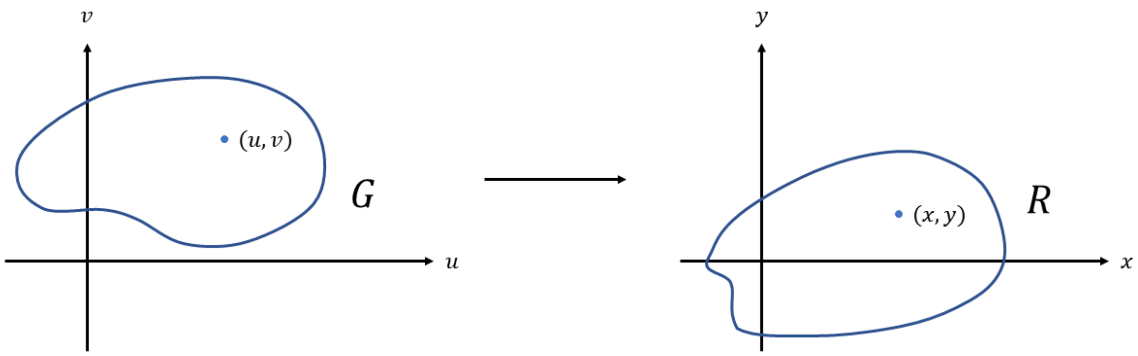
1. 치환적분의 개요

Q: 1변수 함수의 적분에서 사용했던 치환적분법을 다중적분에서도 사용할 수 있는가?

uv-place 의 영역 G가 다음의 변환에 의해 xy-plane 의 영역 R로 옮겨진다.

$$x = g(u,v)$$

$$y = h(u,v)$$



R 위에서 정의된 함수 $F(x,y)$에 대해서 G 위에서 정의된 함수 $F(g(u,v),h(u,v))dudv$ 는 어떤 관계가 있을까?

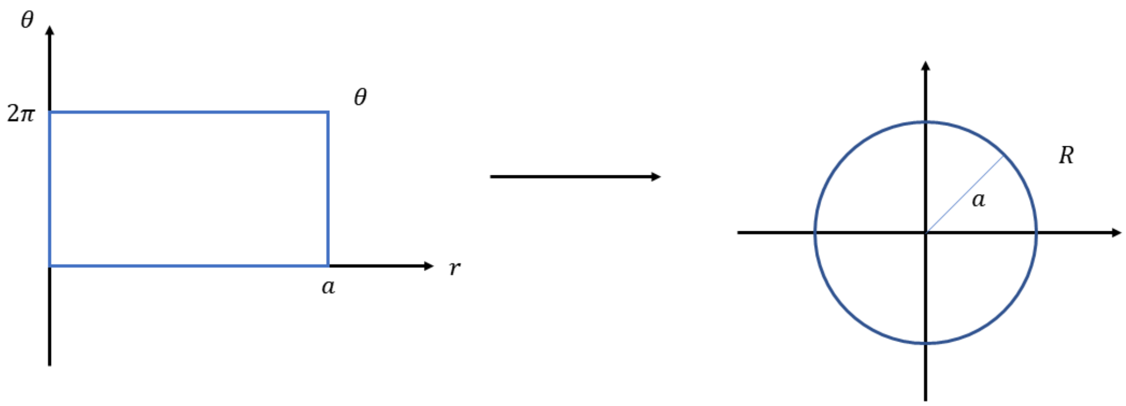
예제)

$$R = \{(x,y) : x^2 + y^2 \leqq a^2\} \qquad a > 0$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$G = \{(r,\theta) : 0 \leqq r \leqq a, 0 \leqq \theta \leqq 2\pi\}$$



$$\iint_R F(x,y)dxdy \cdots\cdots \text{(어떤 관계?)} \cdots\cdots \iint_{[0,a]\times[0,2\pi]} f(r\cos\theta, r\sin\theta)drd\theta$$

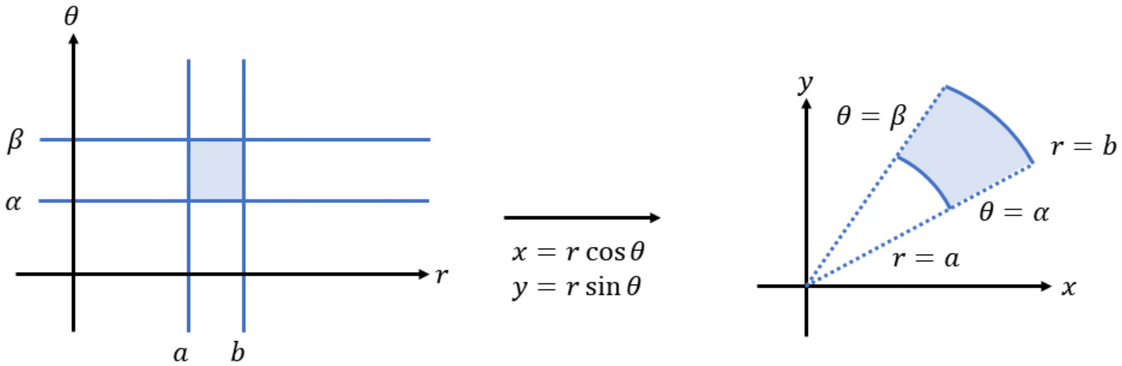
$$\iint_R F(x,y)dxdy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} F(x,y)dydx$$

$dxdy$와 $drd\theta$의 관계를 찾아내면 된다.

1변수의 예

$$\int_{y=a}^{y=b} f(y)dy \qquad y=h(x)$$

$$= \int_{\alpha}^{\beta} f(h(x))\left(\frac{dy}{dx}\right)dx \qquad [\alpha,\beta] \overset{h}{\to} [a,b]$$



$$\text{면적} = \triangle r \triangle\theta = (b-a)(\beta-\alpha)$$

$$\to \frac{1}{2}b^2(\beta-\alpha) - \frac{1}{2}a^2(\beta-\alpha)$$

$$= \frac{1}{2}(\beta-\alpha)(b^2-a^2)$$

$$= \frac{1}{2}(\beta-\alpha)(b-a)(b+a)$$

$$\approx a\triangle r\triangle\theta$$

$$\triangle x\triangle y \approx r\triangle r\triangle\theta$$

$$dxdy = |J(u,v)|dudv$$

(1) 치환적분법 Substitution Rule

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}=\frac{\partial(x,y)}{\partial(u,v)}$$

Jacobian determinant of $x=h(u,v)$

$y=g(u,v)$

$$\iint_R F(x,y)dxdy=\iint_G F(h(u,v),g(u,v))|J(u,v)|dudv$$

*야코비 행렬식에 절대값을 취해야하는 이유는 변환을 취할 때에 오리엔테이션이 바뀔 수 있기 때문이다. 그 경우에는 야코비 행렬식은 음수의 값을 갖는다.

예제)

$$x=r\cos\theta\ ,\ \ y=r\sin\theta$$

$$J(r,\theta)=\begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial\theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial\theta} \end{vmatrix}$$

$$=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r$$

예제)

$$R=\{(x,y):x^2+y^2\leqq a^2\}$$

$$\iint_R dxdy=?$$

$$x=r\cos\theta \qquad y=r\sin\theta$$

$$G=\{(r,\theta):0\leqq r\leqq a, 0\leqq \theta\leqq 2\pi\}$$

$$\iint_G |J(r,\theta)|drd\theta=\iint_G rdrd\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} a^2 d\theta$$

$$= \frac{1}{2} a^2 (2\pi) = \pi a^2$$

연습문제

$x = r\cos\theta$, $y = \frac{r}{\sqrt{2}}\sin\theta$ 를 이용하여 $R = \{(x,y) : x^2 + 2y^2 \leq 4\}$에 대해

$\iint_R 8 - 2x^2 - 4y^2 dxdy$ 를 $r\theta$에 대한 적분으로 치환하여 계산하시오.

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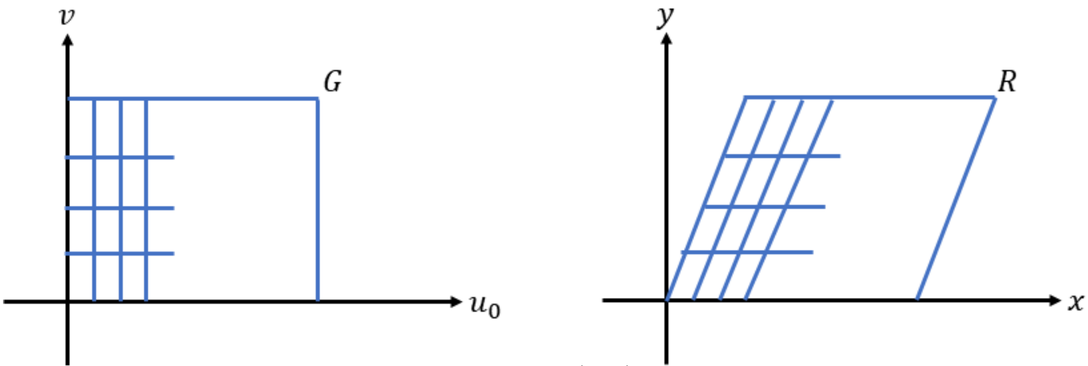
(2) 치환적분법의 원리가 작동하는 이유에 대한 설명

=> 변환을 취할 경우 영역의 면적 변화를 야코비 행렬식으로 어떻게 측정하는가?



$$x = g(u,v)$$

$$y = h(u,v)$$

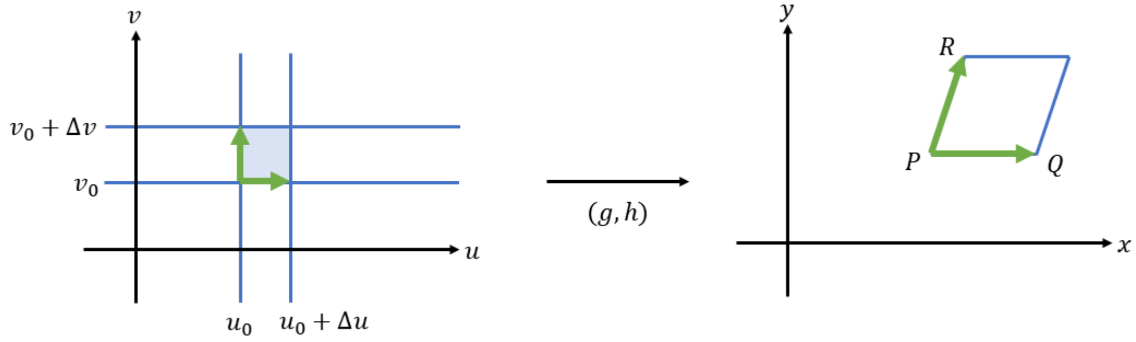
$$\iint_R f(x,y) dxdy = \iint_G f(g(u,v), h(u,v)) |J(g,h)| dudv$$

$$dA = |\overrightarrow{PQ} \times \overrightarrow{PR}|$$

$$P = (g(u_0, v_0), h(u_0, v_0))$$

$$Q = (g(u_0 + \triangle u, v_0), h(u_0 + \triangle u, v_0))$$

$$R = (g(u_0, v_0 + \triangle v), h(u_0, v_0 + \triangle v))$$



$$\overrightarrow{PQ} = (g(u_0 + \triangle u, v_0) - g(u_0, v_0), h(u_0 + \triangle u, v_0) - h(u_0, v_0))$$

$$\approx \left(\frac{\partial g}{\partial u}\triangle u, \frac{\partial h}{\partial u}\triangle u\right)$$

$$\overrightarrow{PR} = (g(u_0, v_0 + \triangle v) - g(u_0, v_0), h(u_0, v_0 + \triangle v) - h(u_0, v_0))$$

$$\approx \left(\frac{\partial g}{\partial v}\triangle v, \frac{\partial h}{\partial v}\triangle v\right)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ g_u \triangle u & h_u \triangle u & 0 \\ g_v \triangle v & h_v \triangle v & 0 \end{vmatrix} = (g_u h_v - h_u g_v) \triangle u \triangle v$$

$$dA \approx |g_u h_v - h_u g_v| \triangle u \triangle v$$

$$\det\begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix} = J(g, h)$$

$$x = g(u, v) \approx g(u_0, v_0) + g_u(u - u_0) + g_v(v - v_0)$$

$$y = h(u, v) \approx h(u_0, v_0) + h_u(u - u_0) + h_v(v - v_0)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} (u_0\ v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix}$$

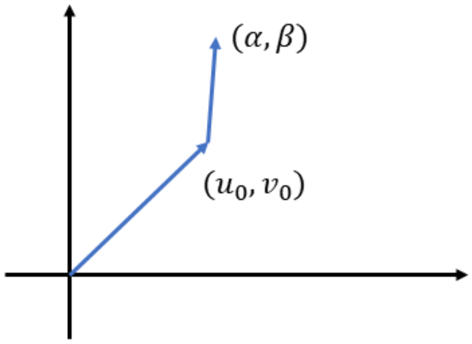
$$new \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \qquad new \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

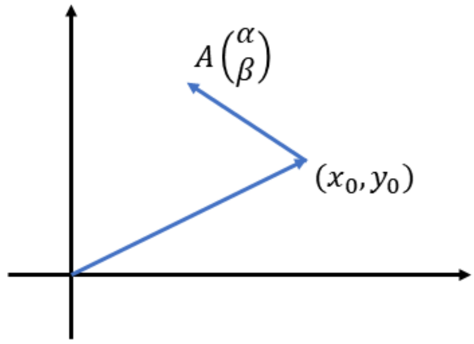
In particular $(u_0, v_0) = (0, 0)$

$(x_0, y_0) = (0, 0)$

$$\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} g_u(0,0) & g_v(0,0) \\ h_u(0,0) & h_v(0,0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

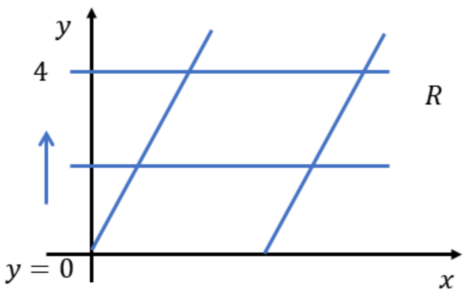
$$\begin{bmatrix} u_0 + \alpha \\ v_0 + \beta \end{bmatrix} \to \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} (u_0, v_0) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$





예제)

$$\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2} dxdy$$



$$y = 2x$$

$$y = 2(x-1)$$

$$u = \frac{2x-y}{2} \qquad v = \frac{y}{2}$$

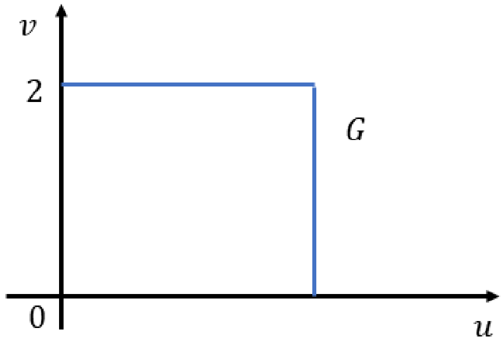
$$0 \leqq y \leqq 4 \qquad 0 \leqq v \leqq \frac{4}{2} = 2 \qquad \begin{cases} x = u+v \\ y = 2v \end{cases}$$

$$\frac{y}{2} \leqq x \leqq \frac{y}{2}+1$$

$$v \leqq x \leqq v+1$$

$$v \leqq u+v \leqq v+1 \qquad u = x-v$$

$$0 \leqq u \leqq 1 \qquad u+v = x$$



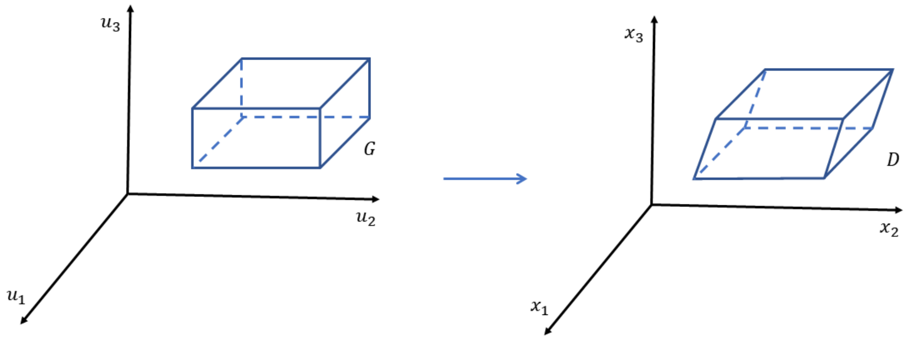
$y=0$		$v=0$
$y=4$	$2x-y=0$	$v=2$
$x=\frac{y}{2}$	$2x-y=2$	$u=0$
$x=\frac{y}{2}+1$		$u=1$

$$\iint_R \frac{2x-y}{2}dxdy = \iint_G u\cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\iint_G u\cdot 2dudv = \int_{v=0}^{v=2}\int_{u=0}^{u=1} 2ududv$$

2. Triple integral의 치환적분



$$X = F(U)$$

$$\iiint_D h(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \iiint_G h \cdot \quad F(u) |J(F)| du_1 du_2 du_3$$

$$X = (x_1, x_2, x_3)$$

$$x_j = F_j(u_1, u_2, u_3)$$

$$J(F) = \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{vmatrix} = \det\left(\frac{\partial F_i}{\partial u_j}\right)$$

$$= \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)}$$

예제)

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 dy dx$$

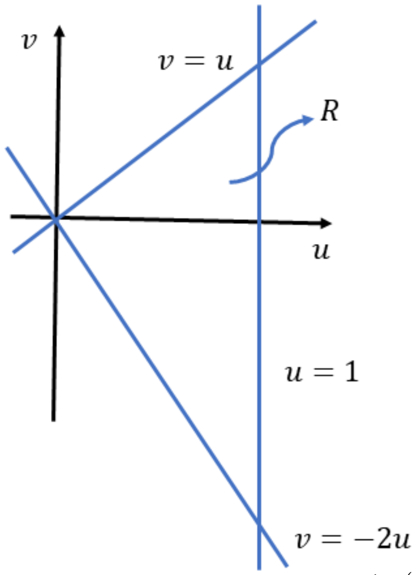
$$u = x + y$$

$$v = y - 2x$$

$$x+y=1 \rightarrow u=1$$

$$y=0 \rightarrow u=x, v=-2x \quad v=-2u$$

$$x=0 \rightarrow u=y, v=y \quad u=v$$



$$\iint_G \sqrt{x+y}\,(y-2x)^2 dxdy = \iint_R \sqrt{u}\,v^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$$x = \frac{u-v}{3}, \quad y = \frac{2u+v}{3}$$

$$\int_0^1 \int_{-2u}^{u} \frac{1}{3}\sqrt{u}\,v^2 dvdu$$

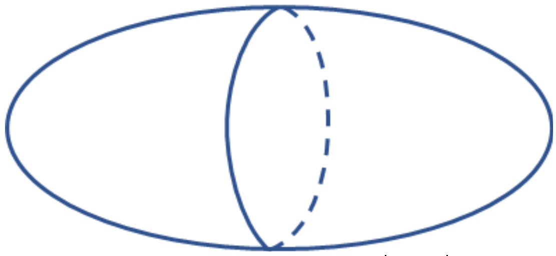
$$= \int_0^1 \frac{1}{3}\sqrt{u}\,\frac{1}{3}v^3 \Big|_{v=-2u}^{v=u} du$$

$$= \int_0^1 \frac{1}{3}\sqrt{u}\,\frac{1}{3}(u^3+8u^3)du$$

$$= \frac{1}{9}\int_0^1 u^3\sqrt{u}\,du$$

예제) Volume of ellipsoid (삼중적분을 이용하여 계산)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$x = au \qquad J(u,v,w) = \begin{vmatrix} a & & \\ & b & \\ & & c \end{vmatrix} = abc$$

$$y = bv$$

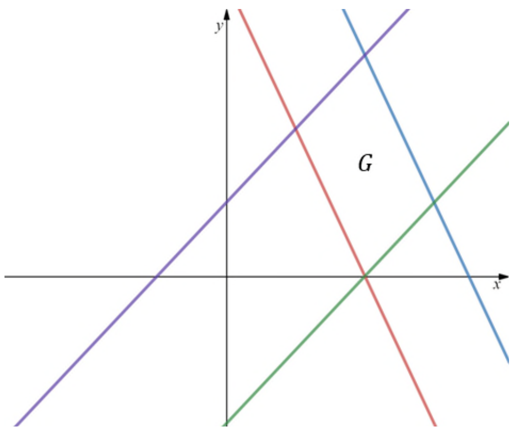
$$z = cw$$

$$V = \iiint_G dxdydz = \iiint_{\{u^2+v^2+w^2<1\}} abc\,dudvdw$$

$$= abc \cdot \quad Vol\,(unit\,sphere)$$

$$= abc\frac{4}{3}\pi$$

예제) 영역 G가 다음 네 개의 직선으로 둘러싸여 있다: $y=-2x+4$, $y=-2x+7$, $y=x-2$, $y=x+1$



중적분 $\iint_G 2x^2 - xy - y^2 dxdy$을 계산하여라. 치환적분을 사용할 것. 주어진 영역 G를 직사각형으로 치환할 수 있는가?

경계를 정의하는 식을 새로운 변수로 두면 된다. 즉

$$v = y + 2x$$

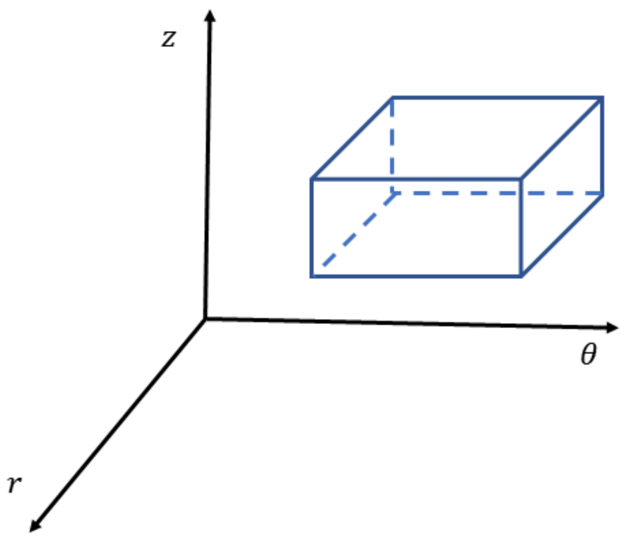
$$u = y - x$$

$$4 \leq v \leq 7$$

$$-2 \leq u \leq 1$$

(연습문제) 치환적분의 계산을 완성하여라.

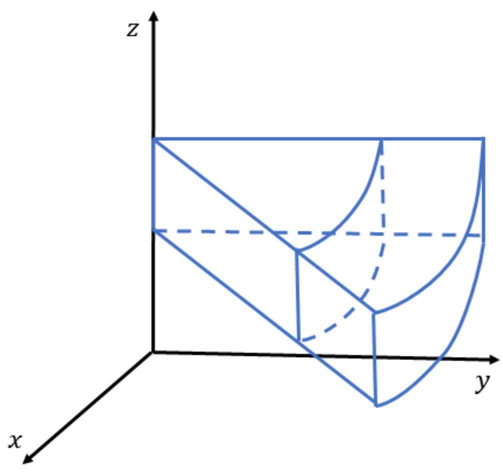
3. 직교좌표를 cylindrical coordinate로 치환할 경우



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$



$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

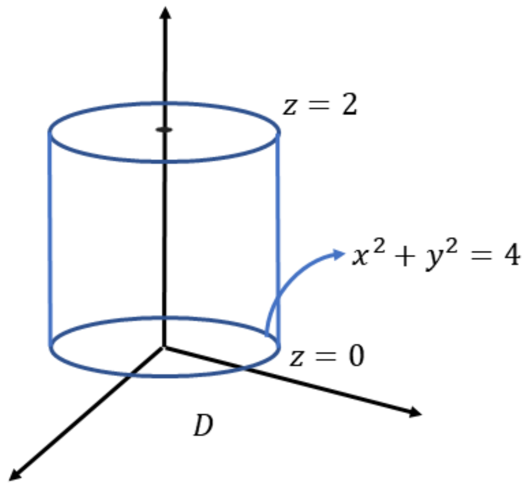
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$dxdydz = rdrd\theta dz$$

예제)

$z=0,\ z=2,\ x^2+y^2=4$ 로 둘러싸인 영역의 부피를 구하라



$$\iiint_D dxdydz$$

$$x = r\cos\theta$$

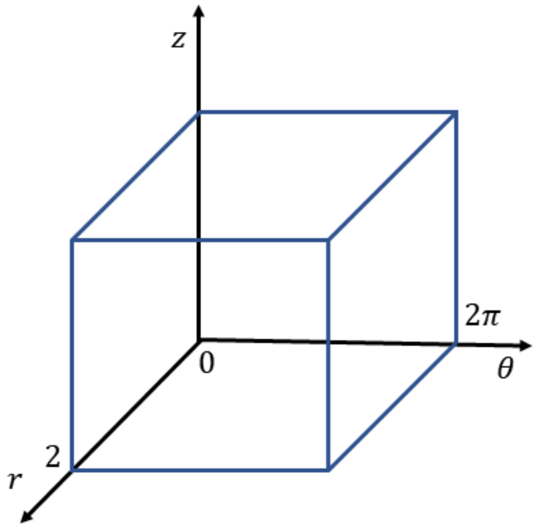
$$y = r\sin\theta$$

$$z = z$$

$z=0$ $\quad$ $z=0$ $\quad$ $0 \leqq r \leqq 2$

$z=2$ $\quad$ $z=2$

$0 \leqq x^2+y^2 \leqq 4$ $\quad$ $0 \leqq r^2 \leqq 4$



$$\iiint_{[0,2]\times[0,2\pi]\times[0,2]} r\,dr\,d\theta\,dz$$

$$=\int_{z=0}^{z=2}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} r\,dr\,d\theta\,dz$$

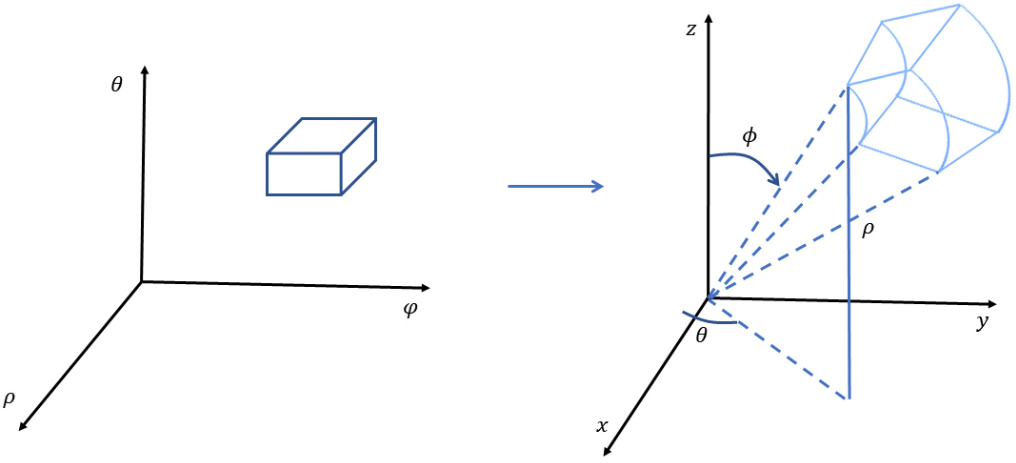
$$=\int_0^2\int_0^{2\pi}\frac{1}{2}r^2\Big|_{r=0}^{r=2}d\theta\,dz$$

$$=\int_0^2\int_0^{2\pi}\frac{1}{2}(2)^2 d\theta\,dz$$

$$=2\int_0^{2\pi}d\theta\int_0^2 dz$$

$$=(2)(2\pi)(2)=8\pi$$

4. Spherical coordinates로 치환하기



$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

야코비행렬식의 계산

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \rho\cos^2\phi\rho\sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \rho\sin^2\phi\rho\sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

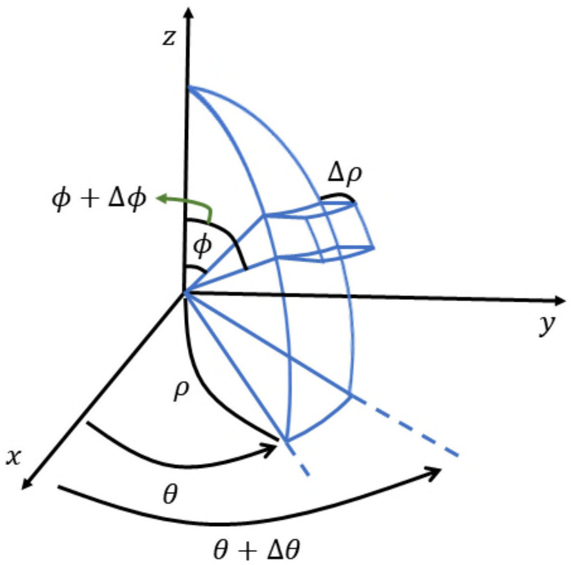
$$= \rho^2\sin\phi$$

치환적분식

$$\iiint_R F(x,y,z)\,dxdydz$$

$$= \iiint_G F(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\rho^2\sin\phi|\,d\rho d\phi d\theta$$

(1) Spherical coordinates 의 volume element 를 이해하기



volume element = $\rho\triangle\phi\cdot\ \ \rho\triangle\theta\sin\phi\cdot\ \ \triangle\rho$

$$=\rho^2\sin\phi\triangle\rho\triangle\phi\triangle\theta$$

예제) 반지름이 1인 구에서 도려낸 cone $\phi=\frac{\pi}{3}$의 부피를 구하여라.

먼저, ρ,ϕ,θ의 범위를 구하여라.

$$0\leqq\ \rho\leqq\ 1,\ 0\leqq\ \phi\leqq\ \frac{\pi}{3},\ 0\leqq\ \theta\leqq\ 2\pi$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1\rho^2\sin\phi d\rho d\phi d\theta$$

일반적으로 ρ, ϕ, θ의 순서로 적분한다.

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\frac{\rho^3}{3}\Big|_0^1 \sin\phi d\phi d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\frac{1}{3}\sin\phi d\phi d\theta$$

$$=\int_0^{2\pi}-\frac{1}{3}\cos\phi\Big|_0^{\frac{\pi}{3}} \mathrm{d}\theta$$

$$=\int_0^{2\pi}\left(-\frac{1}{3}\right)\left(\frac{1}{2}-1\right)d\theta$$

$$=\frac{1}{6}2\pi$$

$$=\frac{\pi}{3}$$

연습문제) 구면 좌표계를 사용하여 cone $z=\sqrt{x^2+y^2}$ 위에 있고 구면 $x^2+y^2+z^2=z$ 아래 있는 영역의 부피를 구하여라

연습문제)H는 xy plane 위에 있고 $x^2+y^2+z^2=1$ 아래 있는 반구에 해당한다.
$\iiint_H (x^2+y^2)dV$를 계산하라.

연습문제) Cylinderical coordinates 를 이용하여
$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{\frac{3}{2}}dzdydx$를 계산하라.

연습문제) Spherical coordinates 를 이용하여
$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)dzdxdy$ 를 계산하라.