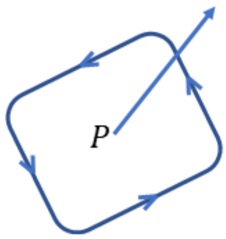
Module Stokes theorem

1. 스톡스의 정리

(1) R^3에서 circulation density를 정의하기



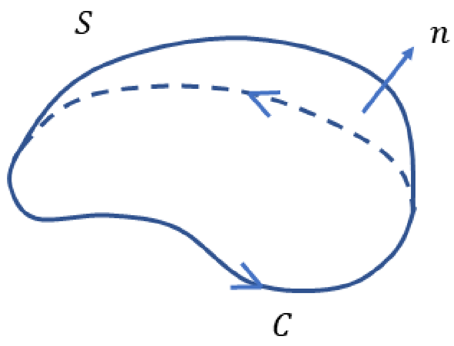
$$F = F_1 i + F_2 j + F_3 k$$

$$curl\ F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

$$= \nabla \times F$$

(2) Stokes의 정리



S : oriented piece wise - smooth surface

$C = \partial S$: simple closed piece wise - smooth boundary curve with positive orientation.

F : S를 포함하는 영역에서 정의된 C' vector field

$$\int_C F \cdot dr = \iint_S curl\, F \cdot n d\sigma$$

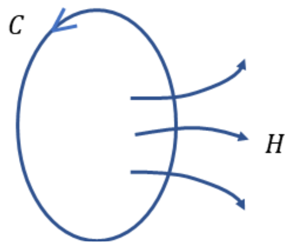
예제)

$S: y+z=2,\ x^2+y^2 \leq 1$

$F = -y^2 i + xj + zk$

$\int_{\partial S} F \cdot dr = ?$

2. Faraday의 법칙



voltage around C = - rate of change of magnetic flux through S

$$\nabla \times E = -\frac{\partial H}{\partial t}$$

$$\text{use } \int_C E \cdot dr = \iint_S (\nabla \times E) \cdot n d\sigma - \frac{\partial}{\partial t} \iint_S H \cdot n dr$$

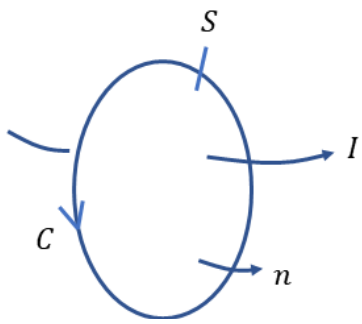
$$= \iint_S -\frac{\partial n}{\partial t} \cdot d\sigma$$

$$= \iint_S (\nabla \times E) \cdot n d\sigma$$

$$= \int_C E \cdot dr$$

Maxwell equation ⇒ Faraday 의 법칙

3. Ampère의 법칙



B : magnetic field

I : current

J : Current density

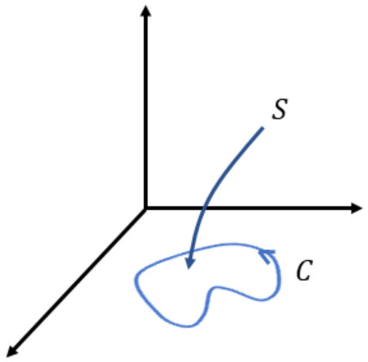
$$\nabla \times B = \frac{4\pi}{C} J$$

$$\Rightarrow \oint_C B \cdot dr = \iint_S \nabla \times B \cdot n d\sigma$$

$$= \iint_S \frac{4\pi}{C} J \cdot n d\sigma$$

$$= \frac{4\pi}{C} I$$

4. “Green 정리와의 연관성”



$$\vec{F} = P\vec{i} + Q\vec{j} \text{ and } \vec{F} = (Q_x - P_y)\vec{k}$$

$$\vec{n} = \vec{k}$$

$$curl\,\vec{F} \cdot \vec{n} = Q_x - P_y \qquad d\sigma = dxdy$$

$$\iint_S curl\,\vec{F} \cdot \vec{n} d\sigma = \iint_S Q_x - P_y)dxdy$$

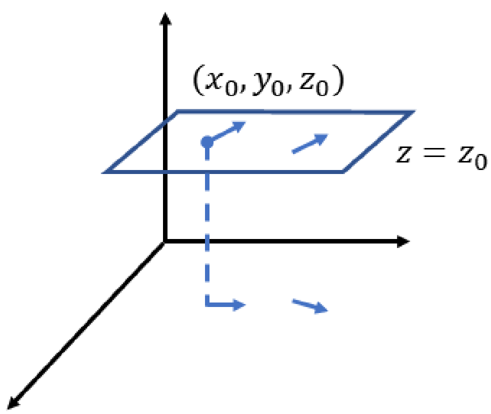
$$= \oint_C Pdx + Qdy$$

5. "Curl의 의미“

Circulation을 각 좌표평면에 projection vector로 표현

$\vec{F}$ xy평면에 대한 circulation (Karis에 대한)

$$\left(proj\,\vec{F}\right)_{xy} = P(x,y,z_0)\vec{i} + Q(x,y,z_0)\vec{j}$$



Circulation density $= (Q_x - P_y)$

→ rotation about $\vec{k}$ direction

→ curl $\vec{F}$의 $\vec{k}$ component

예제)

$$\vec{F} = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$$

C : Curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$

orientation : counter clock wise from above

$\int_C \vec{F}\cdot \ dr = ?$

Sol)

$$\int_C -y^2dx + xdy + z^2dz$$

$$\begin{cases} y+z=2 \\ x^2+y^2=1 \end{cases} \qquad \begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 2-\sin t \end{aligned}$$

$$dz = -dy$$

$$\int_C -y^2dx + xdy - z^2dy$$

$$= \int_C -y^2dx + (x-(2-y^2))dy$$

$$= \int_{t=0}^{t=2\pi} \left[-(\sin^2 t)(-\sin t) + (\cos t - (2-\sin t)^2)\cos t\right]dt$$

$$= \int_0^{2\pi} \cos^2 t - \cos t(2-\sin t)^2 dt$$

$$= \int_0^{2\pi} \cos^2 t + 4\sin t\cos t dt = \int_0^{2\pi} \cos^2 t dt$$

use stokes theorem

$$\int_C \vec{F}\cdot \ dr = \iint_S (\nabla \times \vec{F})\cdot \ \vec{n} d\sigma$$

$$S: \{y+z=2\} \cap \{x^2+y^2<1\}$$

$$\nabla \times \vec{F} = (1+2y)\vec{k}$$

$$z = 2-y \qquad f = 2-y$$

$$\frac{\vec{n} = (-f_x, -f_y, 1)}{\sqrt{f_x^2+f_y^2+1}} \qquad d\sigma = \sqrt{f_x^2+f_y^2+1}\,dxdy$$

$$\iint_D (1+2y)\vec{k}\cdot \ (-f_x, -f_y, 1)dxdy \qquad where\ D=\{x^2+y^2<1\}$$

$$=\iint_D (1+2y)dxdy$$

use polar coordinates

$$x=r\cos\theta,\ y=r\sin\theta$$

$$\int_0^{2\pi}\int_0^1 (1+r\sin\theta)rdrd\theta$$

$$=2\pi\int_0^1 rdr+\int_0^1 r^2dr\int_0^{2\pi}\sin\theta d\theta$$

$$=(2\pi)\frac{1}{2}+0$$

$$=\pi$$

예제)

$$\vec{F}=yz\vec{i}+xz\vec{j}+xy\vec{k}$$

$$S: \{x^2+y^2+z^2=4\}\cap\ \{x^2+y^2<1\}\cap\ \{z>0\}$$

$$\iint_S (\nabla\ \times\vec{F})\cdot\ \ \vec{n}d\sigma=?$$

Stokes : $\iint_S (\nabla\ \times\vec{F})\cdot\ \ \vec{n}d\sigma=\int_C \vec{F}\cdot\ \ ds$

C :위에서 보았을 때 반시계 방향

$$x^2+y^2=1\ \ z=\sqrt{3}$$

$$\vec{r}(t)=(\cos t, \sin t, \sqrt{3})$$

$$\int_C yzdx+xzdy+xydz$$

$$=\int_0^{2\pi}\sqrt{3}\sin tdx+\sqrt{3}\cos tdy$$

$$=\int_0^{2\pi}\sqrt{3}\sin t(-\sin t)dt+\sqrt{3}\cos t(-\cos t)dt$$

$$=\sqrt{3}\int_0^{2\pi}\cos(2t)dt=0$$

6. 전자기학 문제의 응용

Maxwell's equation

$\overrightarrow{E}$: electric field

$\overrightarrow{H}$: magnetic field

ρ: charge density

$\overrightarrow{J}$: current density

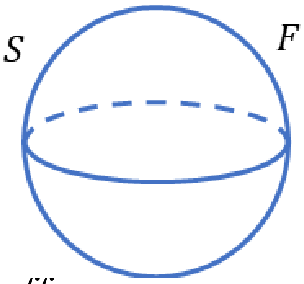
$\nabla \cdot \overrightarrow{E} = \rho$ Gauss's law

$\nabla \cdot \overrightarrow{H} = 0$ no magnetic source

$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{H}}{\partial t}$ Faraday's law

$\nabla \times \overrightarrow{H} = \overrightarrow{J} + \frac{\partial \overrightarrow{E}}{\partial t}$ Ampère's law

Theorem



$$\iint_S curl\, F \cdot\ n d\sigma = 0$$

<u>proof</u>

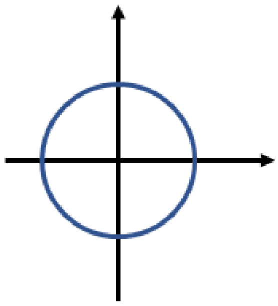
$$\iint_{S_+} curl\, F \cdot\ n d\sigma + \iint_{S_-} curl\, F \cdot\ n d\sigma$$

$$= \int_C F \cdot\ dr + \oint_{-C} F \cdot\ dr - \oint_C F \cdot\ dr = 0$$

$$\int_C F \cdot\ dr = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_{t=0}^{t=2\pi} F(r(t)) \cdot\ r'(t) dt$$

$$\int_{-C} F \cdot\ dr = \int_{t=0}^{t=2\pi} F(r_-(t)) \cdot\ r_-'(t) dt$$



$$r(t) = (\cos t, \sin t)$$

$$r_-(t) = (\cos(-t), \sin(-t)) = r$$

$$\int_0^{-2\pi} F(r_-(-s)) \cdot\ (-r'(s)) ds$$

$$r_-'(t) = -r'(-t) \qquad r_-' = (-s) = -r'(s)$$

$$r_-(t) = r(2\pi - t)$$

$$r_-'(t) = -r'(2\pi - t)$$

$$\int_{-C} = \int_0^{2\pi} F\cdot \ (r_-(t))\cdot \ r_-'(t)dt$$

$$= \int_{2\pi}^{0} F(r(s))(-1)r'(s)(-1)ds$$

$$=-\int_0^{2\pi} F(r(s))\cdot \ r'(s)ds$$

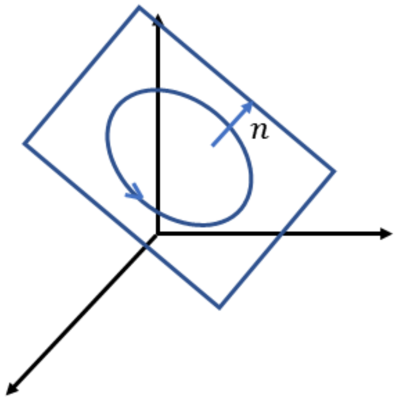
예제)

$$\int_C F\cdot \ dr$$

$$F=-y^2 i+xj+z^2 k$$

$$C: \ \{y+z=2\}\cap \ \{x^2+y^2=1\}$$

위에서 아래로 봤을 때 반시계 방향으로



$$n=\frac{(0,1,1)}{\sqrt{2}}$$

$$\iint_S curlF\cdot \ nd\sigma$$

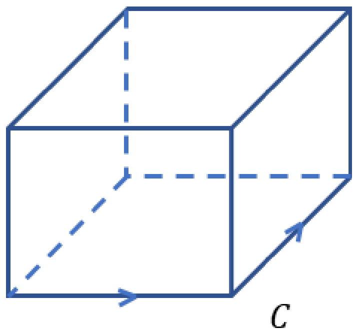
$$curl\ F=\begin{vmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}=-f_u i-f_v j+k$$

$$-curl\ F\cdot \ nd\sigma$$

$$=curl\ F\cdot \ (-f_u i-f_v j+k)dudv$$

$$=(1+2v)dudv$$

예제)



S :top + 4 sides

$$\iint_S curl\ F\cdot\ \ n d\sigma$$

$$= \iint_{bottom} curl\ F\cdot\ \ n d\sigma$$

$$= \oint_C F\cdot\ \ dr$$

$$F = xyzi + xyj + x^2yzk$$

$$z = -1 \quad \iint_{[-1,1]\times[-1,1]} (-xy, xy, -x^2y)$$

$curl\ F = ?$

$n = k$

예제) rectangular region

$R = [1,2]\times[3,4]$

$$\iint_R 2x^2 + 3y^2 dxdy$$

Fubin: $\int_3^4 \int_1^2 2x^2 + 3y^2 dxdy$

$$= \int_1^2 \int_3^4 2x^2 + 3y^2 dydx$$

예제)

$$\iint_R x\cos(xy) dxdy$$

$$R = [0,1]\times[0,\pi]$$

$$\int_0^1 \int_0^\pi x\cos(xy)\,dy\,dx$$

$$= \int_0^1 \sin(xy)\Big|_{y=0}^{y=\pi} dx$$

$$= \int_0^1 \sin(\pi x) \cdot \quad dx$$