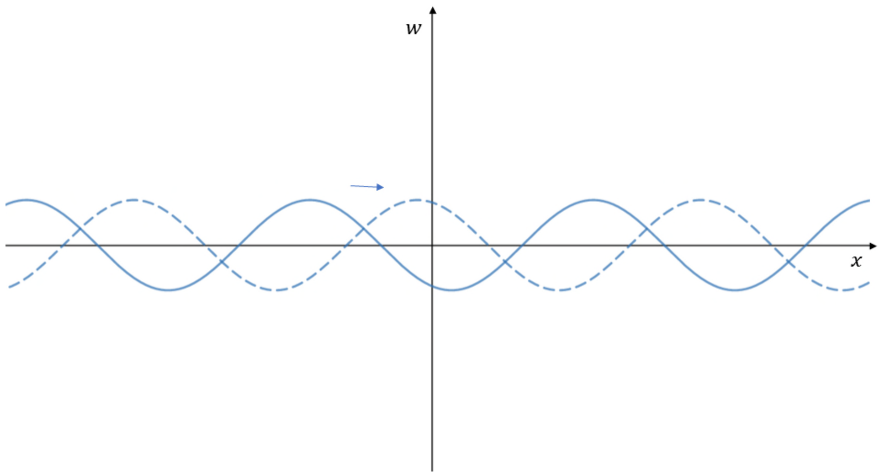
Module Partial derivative

1. 편미분에 대한 동기부여

A. Wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial w^2}{x^2}$$



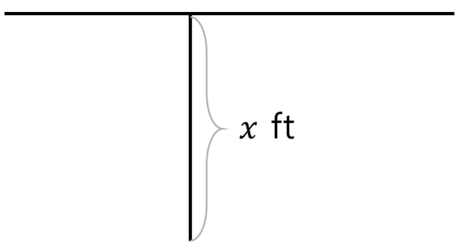
w : 수면의 높이 $w = w(x,t)$

t : 시간

x : 거리에 대한 변수

c :파동의 속도

B. 계절에 따른 지표 아래의 온도변화



$w(x,t)$ t day

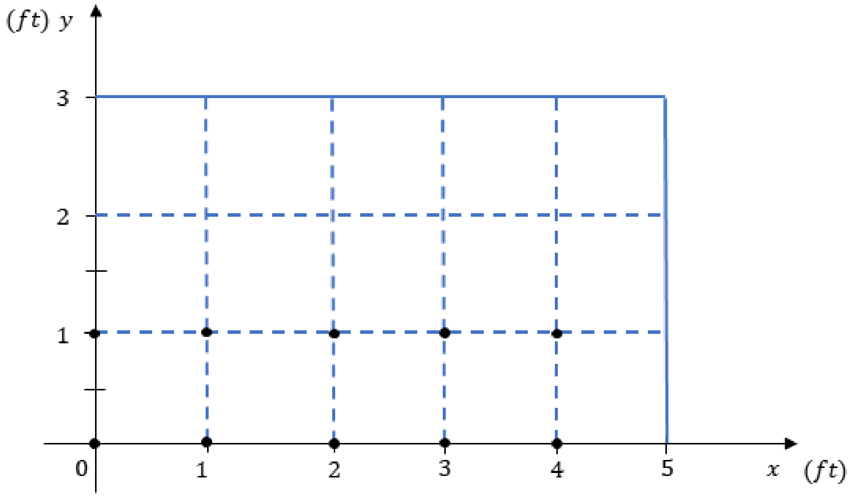
$$w_{xx} = \frac{1}{c^2} w_t$$

$$w(x,t) = \cos(1.7 \times 10^{-2} t - 0.2x) e^{-0.2x}$$

$c = 0.19 \ ft^2/day$ (건조한 땅)

C. 편미분을 이해하는 물리적 Example

The temperature of the metal plate



y \ x	0	1	2	3	4	5
3	85	90	·	·	·	·
2	100	110	120	145	·	·
1	125	128	135	160	·	·
0	120	135	155	160	·	·

단위 (℉)

$$T = T(x,y) \qquad \text{온도}$$

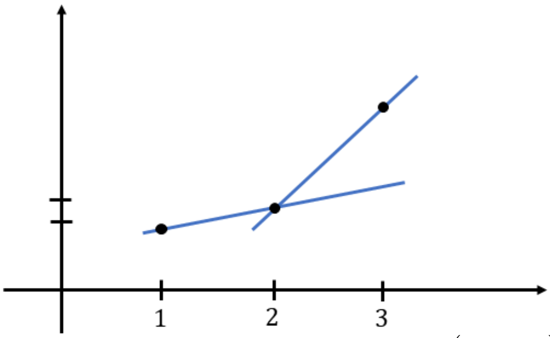
$$\frac{\partial T}{\partial x}(2,1) = \lim_{h \to 0} \frac{T(2+h,1) - T(2,1)}{h}$$

$$\approx \frac{T(3,1) - T(2,1)}{1}$$

$$= 160 - 135 = 25\ ℉ / ft$$

$$\frac{T(2-1,1) - T(2,1)}{(-1)} = \frac{128 - 135}{(-1)}$$

$$= 7\ ℉ / ft$$



$$\frac{\partial T}{\partial y}(2,1) = \lim_{h \to 0} \frac{T(2,1+h) - T(2,1)}{h}$$

$$\approx \frac{T(2,1+1) - T(2,1)}{1} = 120 - 135 = -15$$

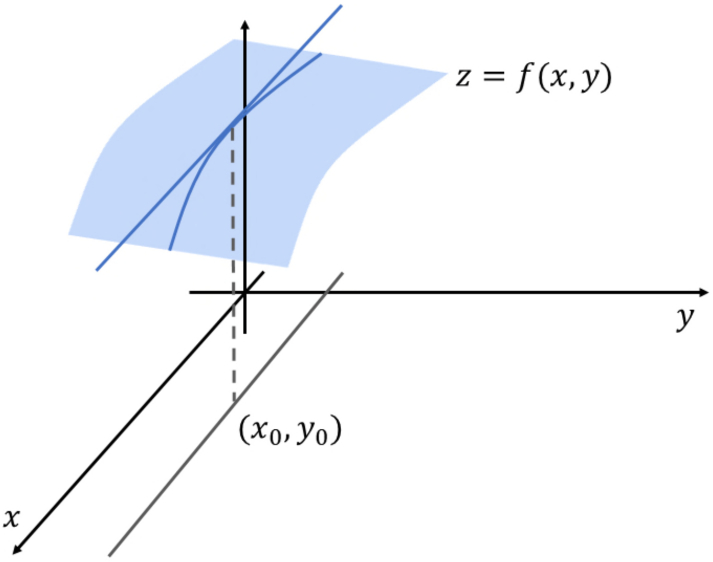
$$\approx \frac{T(2,1-1) - T(2,1)}{(-1)} = \frac{155 - 135}{(-1)} = -20$$

2. 편미분 (Partial derivative)

$$z = f(x, y)$$

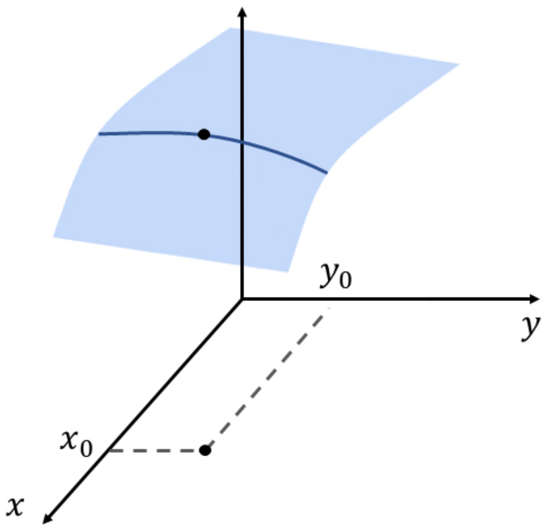
$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$



$z = f(x, y_0)$ 의 그래프

$z = f(x, y_0)$의 $x = x_0$에서의 접선의 기울기 $= \dfrac{\partial f}{\partial x}$



$$z=f(x_0,y) \text{ 의 그래프}$$

$$y=y_0\text{에서의 접선의 기울기 } =\frac{\partial f}{\partial y}$$

예제)

$$f(x,y)=x^2+3xy+y-1$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}x^2+3\frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial x}y-\frac{\partial}{\partial x}1$$

$$=2x+3y(1)+0-0$$

$$=2x+3y$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}x^2+3\frac{\partial}{\partial y}(xy)+\frac{\partial}{\partial y}y-\frac{\partial}{\partial y}1$$

$$=0+3x+1-0$$

$$=3x+1$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,5)}=2(4)+3(5)=8+15=23$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,5)}=3(4)+1=13$$

예제)

$$f(x,y)=\frac{2y}{y+\cos x}$$

$$\frac{\partial f}{\partial x}=2\frac{\frac{\partial y}{\partial x}(y+\cos x)-y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$=2\frac{0-y(0-\sin x)}{(y+\cos x)^2}$$

$$=\frac{2y\ \sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y}=\frac{2\cos x}{(y+\cos x)^2}$$

3. 고계미분 Higher order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad (f_y)_x = f_{yx}$$

예제)

$$f(x,y) = x^2 + y^2$$

$$f_x = 2x, \; f_{xx} = 2$$

$$f_{xy} = 0$$

$$f_y = 2y \quad f_{yy} = 2 \quad f_{yx} = 0$$

예제) 일반적으로 미분의 순서를 바꾸면 서로 미분 값이 같지 않다.

$$f_{xy} \neq f_{yx}$$

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0 \quad , & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0)}{h} = \lim_{h \to 0} \frac{1}{h} \frac{h \cdot \; 0(h^2 - 0^2)}{h^2 + 0^2} = 0$$

$$f_x(x,y) = \frac{x^3 y - x y^3}{x^2 + y^2}$$

$$f_x(x,y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - (x^3 y - x y^3)2x}{(x^2 + y^2)^2}$$

$$= \frac{y\left[(3x^2 - y^2)(x^2 + y^2) - (x^3 - x y^2)2x\right]}{(x^2 + y^2)^2}$$

$$3x^4 + 2x^2 y^2 - y^4 - 2x^4 + 2x^2 y^2$$

$$= 2x^4 + 4x^2 y^2 - y^4$$

$$f_x(x,y) = \frac{y(2x^4 + 4x^2 y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\frac{h(-h^4)}{h^4}=-1$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)}{h}=0$$

$$f_y(x,y)=x\frac{[(x^2-y^2)+y(-2y)](x^2+y^2)-y(x^2-y^2)2y}{(x^2+y^2)^2}$$

$$=x\frac{(x^2-3y^2)(x^2+y^2)-2y^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$=x\frac{x^4-2x^2y^2-3y^4-2x^2y^2+2y^4}{(x^2+y^2)^2}$$

$$=x\left(\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}\right)$$

$$f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)}{h}=\lim_{h\to 0}\frac{1}{h}h\frac{(h^4)}{h^4}=1$$

$$\Rightarrow f_{xy}(0,0)\neq f_{yx}(0,0)$$

Mixed derivative theorem (Euler)

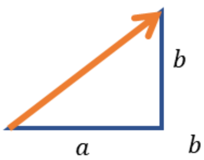
$u\subset R^n$ (u는 대문자로)에서 정의된 이급함수 즉 모든 두 번 미분한 함수들이 연속일 때 $f(x_1,x_2,\ldots,x_n)$ 와 $P\in u$ (u는 대문자로) 에 대해서 $f_{x_ix_j}(P)=f_{x_jx_i}(P)$

오일러 정리의 응용

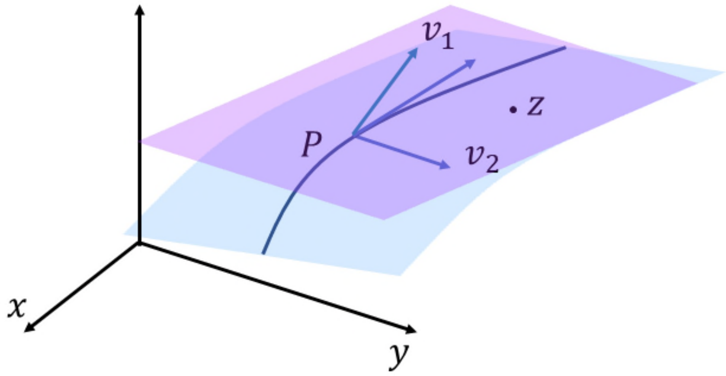
$$\frac{\partial^5}{\partial x^2\partial y^3}(x\sin(y^2)+e^{y^2})$$

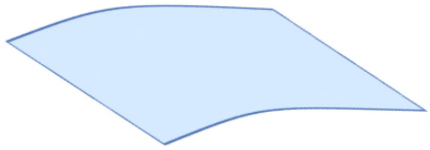
$$\frac{\partial^2}{\partial y\partial x}(x\ln(xy))$$

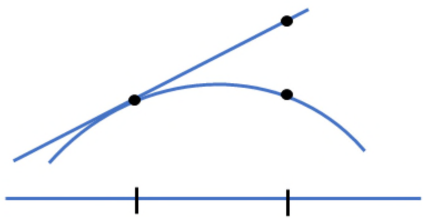
$$(x_0,y_0)f(x_0,y_0)$$



$$\frac{b}{a}=\frac{\partial f}{\partial x}$$







$$v_1 = \left(1,0,\frac{\partial f}{\partial x}\right)$$

$$v_2 = \left(0,1,\frac{\partial f}{\partial y}\right)$$

$$v_1 \times v_2 \quad \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -f_x i - f_y j + \mathrm{k}$$

equation of plane: $-f_x(x-x_0)-f_y(y-y_0)+(z-z_0)=0$

$$z-z_0 = f_x(x-x_0)+f_y(y-y_0)$$

$$\Rightarrow \Delta z \approx f_x \Delta x + f_y \Delta y$$

$$\frac{\Delta z}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$