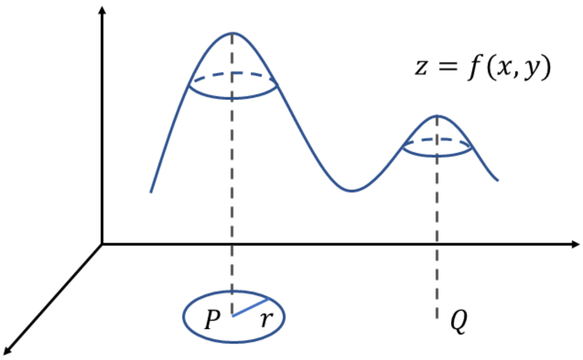
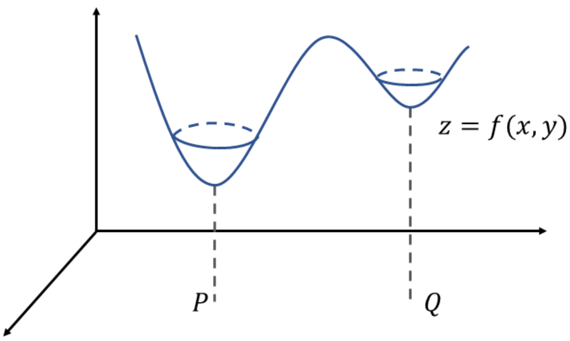
Module Optimization (several variable)



f has local minimum at P and Q

$$f(P) \geqq f(x),\ x \in B_P(r)$$

$$f(Q) \geqq f(x),\ x \in B_Q(s)$$



f has local minimum at P and Q

$$f(P) \leqq f(x),\ x \in B_P(r)$$

$$f(Q) \leqq f(x),\ x \in B_Q(s)$$

정의) P : critical point of f $\Leftrightarrow$ 다음 중 하나를 만족할 경우

① $f_x = f_y = 0$ at P

② f_x or f_y fail to exist at P

<u>임계점 정리</u>

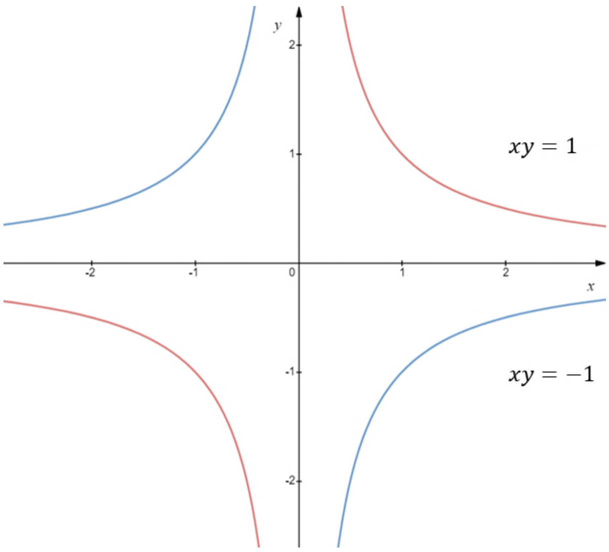
$$U \subset R^2$$

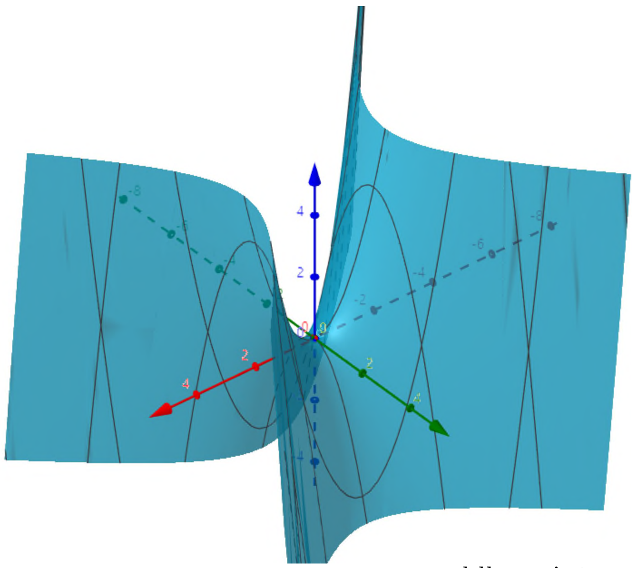
만약 미분가능한 함수f : $u \rightarrow R$ 가 점 P에서 국소적인 최대나 최소가 되면

$$f_x(P) = f_y(P) = 0$$

Remark) 위의 정리의 역은 성립하지 않는다

생각해볼 함수 $z = xy$





saddle point

$$z|_{y=x} = x^2$$

$$z|_{y=-x} = -x^2$$

$$\left.\frac{\partial z}{\partial x}\right|_{(x.y)=(0,0)} = y\Big|_{(x,y)=(0,0)} = 0$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(0,0)} = x\Big|_{(x,y)=(0,0)} = 0$$

Def 함수 f의 Hessian determinant 또는 discriminant (헤세 행렬의 행렬식)

$$H_f = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

■ 극대, 극소의 판별법 P : critical point of f

(1) $H_f(P) > 0,\ f_{xx}(P) > 0$ $\Rightarrow$ P : local min

(2) $H_f(P) > 0,\ f_{yy}(P) < 0$ $\Rightarrow$ P : local max

(3) $H_f(P) < 0,$ $\Rightarrow$ P : saddle point

(4) $H_f(P) = 0,$ 판별할 수 없다.

예제)

$$f(x,y) = x^2 + y^2$$

$$f_x = 2x = 0 \qquad f_y = 2y = 0 \qquad \Rightarrow \qquad (0,0)$$

$$x = 0 \qquad\qquad y = 0$$

$$H_f = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

$f_{xx} = 2 > 0$ $(0,0)$은 f의 극소점

예제)

$$f(x,y) = x^3 - 12xy + 8y^3$$

$$f_x = 3x^2 - 12y = 0 \qquad x^2 = 4y$$

$$f_y = -12x + 24y^2 = 0 \qquad 2y^2 = x$$

$$4y^4 = 4y \qquad y = 0 \quad y = 1$$

$$x = 0 \quad x = 2$$

$$(0,0) \quad (2,1)$$

$$f_{xx} = 6x$$

$$f_{xy} = -12$$

$$f_{yx} = -12$$

$$f_{yy} = 48y$$

$$H_f(0,0) = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -(12)^2 < 0$$

$(0,0)$: saddle point

$$H_f(2,1) = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix} = 12 \cdot \ 48 - 12 \cdot \ 12$$

$$= 12(48-12) > 0$$

$$f_{xx}(2,1) = 6 \cdot \ 2 > 0 \qquad \text{극소점}$$

Second derivative test

f is continuous on a disk centered at (a,b)

$f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist and continuous on the same disk

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ discriminant

① $D > 0$, $f_{xx} < 0$ f has local max

② $D > 0$, $f_{xx} > 0$ f has local min

③ $D < 0$ f has a saddle point

④ $D = 0$ test is inconclusive

Def Saddle point

The point (a,b) is saddle point of f

if (a,b) is critical pt of f on a disk and continous at (a,b). It satisfies that

there are points (x_1, y_1) and (x_2, y_2) in disk

such that $f(x_1,y_1) > f(a,b)$ and $f(x_2,y_2) < f(a,b)$

예제)

$f(x,y) = x^2 - y^2$

$(0,0)$ is saddle point of f

$$f(1,0) = 1 > f(0,0)$$

$$f(0,1) = -1 < f(0,0)$$

$$f_x = 2x = 0, \ f_y = -2y = 0$$

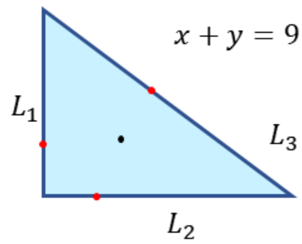
$\Rightarrow$ $(0,0)$ is critical point.

예제)

$$f(x,y)=2+2x+2y-x^2-y^2$$

영역 R은 $x=0, y=0, x+y=9$ 경계로 가는 원

$$\max_R f, \min_R f$$



Sol) 임계점, boundary에서 함숫값 조사

$$f_x = 2-2x = 0 \quad x=1$$

$$f_y = 2-2y = 0 \quad y=1$$

$$f_{xx} = -2 \qquad f_{xy} = 0 \qquad f_{yy} = -2$$

$$H_f = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \quad , \ f_{xx} < 0$$

f has a local max at $(1,1)$

$$f(1,1) = 2+2+2-1-1 = 4$$

$$L_1 : x=0 \qquad f(0,y) = 2+2y-y^2 \qquad 0 \leqq x \leqq 9$$

$$= y(2-y)+2$$

$$f'(0,y) = 2-2y = 0$$

$$y=1$$

$$\max_{L_1} f = 3, \min_{L_1} f = 9 \cdot (-7) = -63+2- = -61$$

$$L_2 : y=0 \qquad f(x,0) = 2-2x-x^2$$

$$\max_{L_2} f = 3, \min_{L_2} f = -61$$

$$L_3 \ : \ f = 2 + 2\cdot \quad 9 - x^2 - (9-x)^2$$

$$= 20 - x^2 - (x^2 - 18x + 81)$$

$$= -2x^2 + 18x - 61$$

$$= -2(x^2 - 9x) - 61$$

$$= -2\left(x - \frac{9}{2}\right)^2$$

$$x = \frac{9}{2}$$

$$f = 20 - 2\cdot \quad \frac{81}{4} = \frac{-41}{2}$$

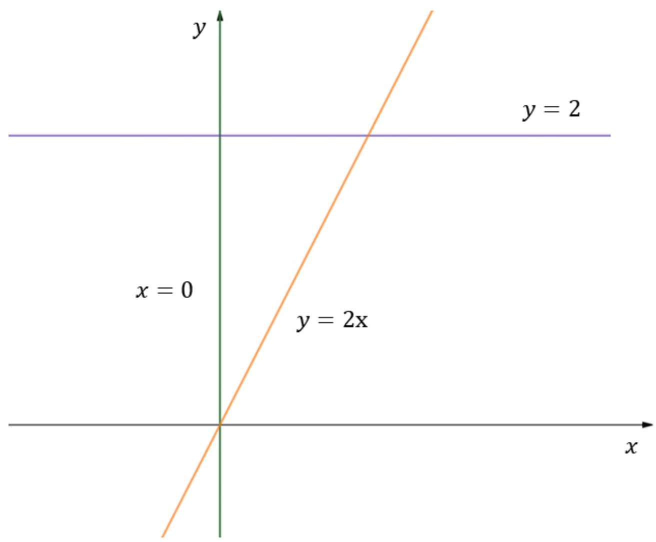
		극 값
경계	내부 end points	$3, -\frac{41}{2}$ $f(0,0) = 2$ $f(0,4) = -61$ $f(4,0) = -61$
내부	$(1,1)$	4

$\max = f = 4$, $\min = -61$

예제)

$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

triangular region enclosed by lines $x = 0, y = 2, y = 2x$



내부:

$$f_x = 4x - 4 = 0 \quad x = 1$$

$$f_y = 2y - 4 = 0 \quad y = 2$$

$$f_{xx} = 4,\ f_{xy} = 0,\ f_y = 2$$

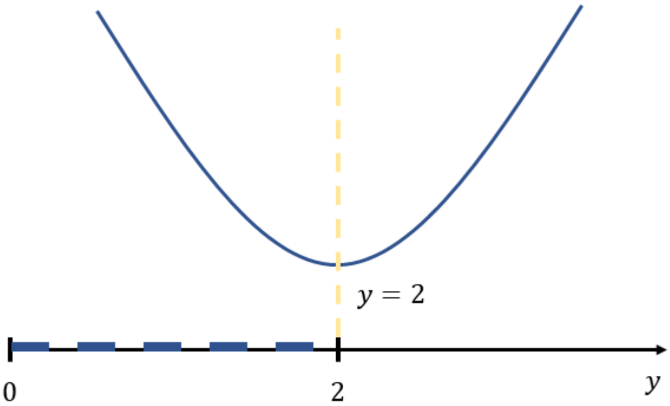
$$D = 8 - 0^2 > 0\ ,\ f_{xx} > 0$$

local min

$$f(1,2) = 2 - 4 + 4 - 8 + 1 = -5$$

$$x = 0 \qquad 0 < y < 2$$

$$f(0,y) = y^2 - 4y + 1 = (y-2)^2 - 3$$



$$y = 2 \qquad f(x,2) = 2x^2 - 4x + 4 - 8 + 1$$

$$= 2(x^2 - 2x + 1) - 5$$

$$= 2(x-1)^2 - 5$$

선 내부에 없음

$$y = 2x \qquad f = 2x^2 - 4x + 4x^2 - 8x + 1$$

$$0 < x < 1 \qquad = 6x^2 - 12x + 1$$

$$= 6(x^2 - 2x + 1) - 5$$

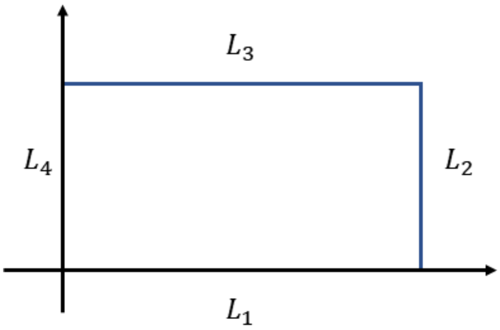
$$= 6(x-1)^2 - 5$$

corners : $(0,0), (0,2), (1,2)$

예제)

$$f(x,y) = x^2 - 2xy + 2y$$

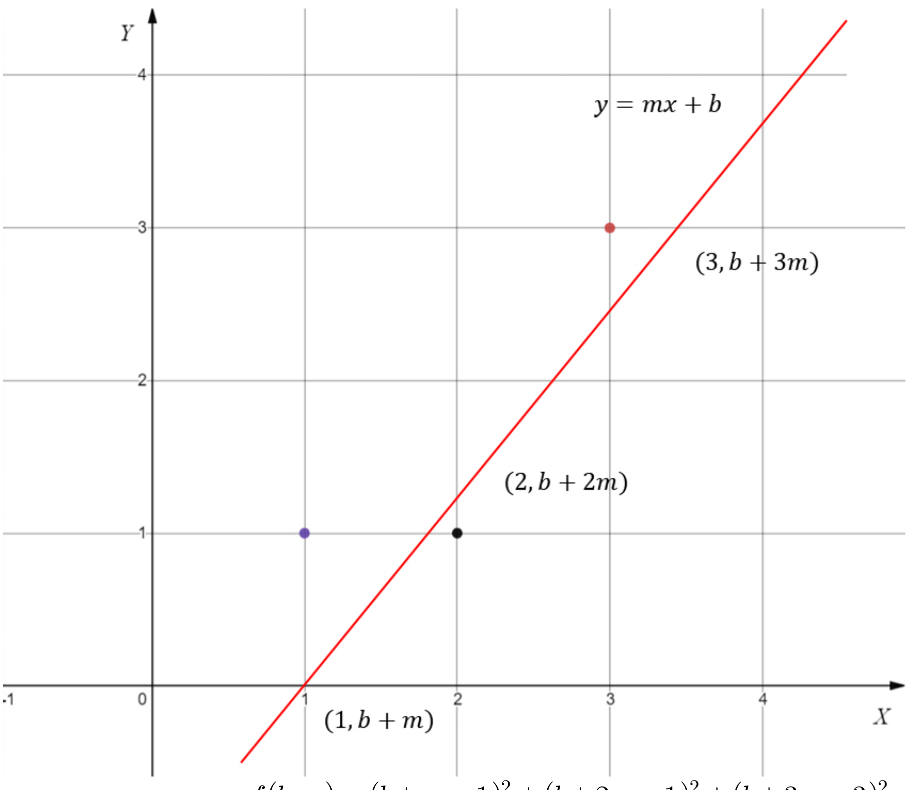
$$R = \{(x,y) |\ 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$$



응용문제

Least squares lines (Regressive line)

Find least square line for data points $(1,1), (2,1), (3,3)$



$$f(b,m) = (b+m-1)^2 + (b+2m-1)^2 + (b+3m-3)^2$$

$$f_b = 2(b+m-1) + 2(b+2m-1) + 2(b+3m-3)$$

$$= 6b + (2+4+6)m - 2 - 2 - 6$$

$$= 6b + 12m - 10$$

$$f_m = -24 + 12b + 28m$$

$$f_k = 0, \; f_m = 0, \; b = -\frac{1}{3}, \; m = 1, \; f_{bb} = 6 > 0$$

$$D = 24$$

$$= f_{bb} f_{mm} - f_{bm}^2$$

$$y = 1 - \frac{1}{3}$$

We want to fit the best line to some data in the plane. We measure the distance from a line to the data points.

The smallest this sum of squares is the better the line fits the data general.

$$f = \sum (mx_j + b - y_j)^2$$

$$= \sum m^2 x_j^2 + b^2 + y_j^2 + 2mbx_j - 2by_j - 2mx_j y_j$$

$$= m^2 \sum x_j^2 + 2mb \sum x_j - 2m \sum x_j y_j - 2b \sum y_j + nb^2 + \sum y_j^2$$

$$f_m = 2m \sum x_j^2 + 2b \sum x_j - 2 \sum x_j y_j = 0$$

$$f_b = 2m \sum x_j - 2 \sum y_j + 2nb = 0$$

$$\begin{bmatrix} A & B \\ B & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum x_j y_j \\ \sum y_j \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{An - B^2} \begin{bmatrix} n & -B \\ -B & A \end{bmatrix} \begin{bmatrix} \sum x_j y_j \\ \sum y_j \end{bmatrix}$$

$$m = \frac{n \sum x_j y_j - \sum x_j \sum y_j}{n \sum x_j^2 - \left(\sum x_j\right)^2}$$

$$= \frac{\sum x_j \sum y_j - n \sum x_j y_j}{\left(\sum x_j\right)^2 - n \sum x_j^2}$$

$$m \sum x_j^2 + b \sum x_j = \sum x_j y_j$$

$$m \sum x_j + nb = \sum y_j$$

$$b = \frac{1}{n} \left(\sum y_j - m \sum x_j\right)$$

$$= \bar{y} - m\bar{x}$$

$$\sigma^2 = \sum (x_j - \bar{x})^2 = \sum x_j^2 - 2\bar{x} \sum x_j + n\overline{x^2}$$

$$= \sum x_j^2 - 2n\overline{x^2} + n\overline{x^2}$$

$$= \sum x_j^2 - n\overline{x^2}$$

$$(\sum x_j)^2 - n\sum x_j^2 = n^2\overline{x^2} - n\sum x_j^2$$

$$= - n\sigma^2$$

$$\frac{n\sum x_j y_j - \sum x_j \sum y_j}{n\sigma^2} = \frac{\frac{\sum x_j y_j}{n} - \frac{\sum x_j}{n}\frac{\sum y_j}{n}}{\frac{\sigma^2}{n}}$$

$$= \frac{\overline{(x\cdot\ \ y)} - \overline{x}\overline{y}}{\frac{\sigma^2}{n}}$$