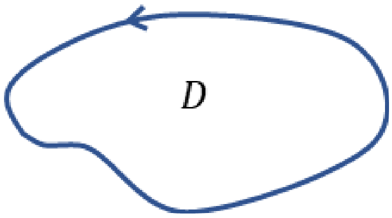
Module Green Theorem

영역 D가 주어져 있고 vector field $F(x,y)$는 D를 포함한 어떤 영역 위에 정의되있다.
∂D를 D의 경계선이라 한다.

정리) Let ∂D be a positively oriented piecewise - smooth, simple closed curve in the plane. Let $F=P(x,y,)i+Q(x,y)j$

Then $\int_{\partial D} Pdx+Qdy=\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$



예제) D : $x=0,x=1,y=0,y=1$로 둘러싸인 rectangle.

$$F=(x^2+y^2)i+(y+x^2)j$$

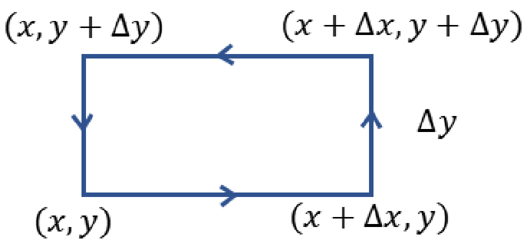
$$\int_{\partial D}(x+y^2)dx+(y+x^2)dy=\iint_D \frac{\partial y+x^2}{\partial x}-\frac{\partial x+y^2}{\partial y}dxdy$$

$$=\iint_D (2x-2y)dxdy$$

$$=\int_0^1\int_0^1 2(x-y)dxdy$$

1. Green의 정리의 증명 (D가 rectangle인 경우)

D를 작은 rectangle들로 분할한다.
다음의 rectangle에 대해 증명한다.



각 line을 따라 F가 한 work을 계산하여 다 더해본다.

Bottom: $F(x,y)\cdot\ i\,\triangle x=P(x,y)\triangle x$

Top: $F(x,y+\triangle y)\cdot\ (-i)\triangle x=-P(x,y+\triangle y)\triangle x$

Right: $F(x+\triangle x,y)\cdot\ j\ \triangle y=N(x+\triangle x,y)\triangle y$

Left: $F(x,y)(-j)\triangle y = -N(x,y)\triangle y$

Top and bottom: $-(P(x,y+\triangle y)-P(x,y))\triangle x \approx -\dfrac{\partial P}{\partial y}\triangle y\triangle x$

Right and left: $(N(x+\triangle x,y)-N(x,y))\triangle y \approx \dfrac{\partial N}{\partial x}\triangle x\triangle y$

(*) Total work along the boundary of rectangle = $\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial P}{\partial y}\right)\triangle x\triangle y$

Summing up this identity over all sub rectangles of the given rectangle. We obtain Green's theorem.

(*)를 circulation around rectangle 이라 부르고 circulation / $\triangle x\triangle y$를 circulation density라 부른다. 우리가 증명한 것은 circulation density at $(x,y) \approx \dfrac{\partial N}{\partial x}-\dfrac{\partial P}{\partial y}$

오른쪽 양을 curve of vector field F라 부르고 curl F로 나타낸다.

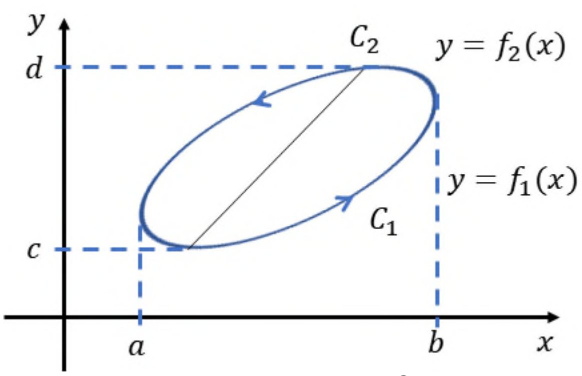
예제) unit circle circulate $F = xi + xyj$

2. Green 정리 확인

$$\int_{t=0}^{t=2\pi}\left(\cos t\frac{dx}{dt}+\cos t\sin t\frac{dy}{dt}\right)dt$$

$$\int_{\{x^2+y^2<1\}}\frac{\partial xy}{\partial x}-\frac{\partial x}{\partial y}dxdy$$

$$=\int_{\{x^2+y^2<1\}}ydxdy$$



$$\int_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}dxdy$$

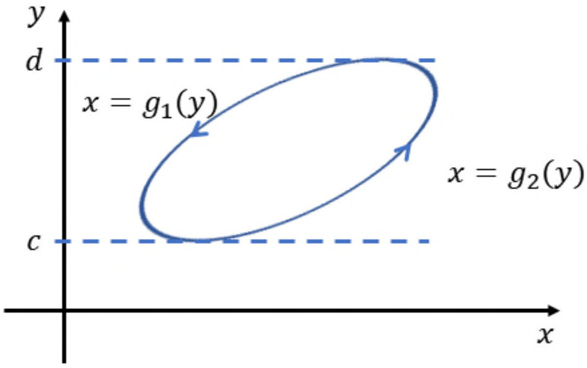
$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dydx$$

$$= \int_a^b P(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b P(x,f_2(x)) - P(x,f_1(x))dx$$

$$= -\int_{C_2} Pdx - \int_{C_1} Pdx$$

$$= -\int_C Pdx$$



$$C_1 \ :(g_1(y),y)$$
$$C_2 \ :(g_2(y),y)$$

$$\int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x} dxdy = \int_c^d Q(g_2(y),y) - Q(g_1(y),y)dy$$

$$= \int_{C_2} Qdy + \int_{C_1} Qdy$$

$$= \int_C Qdy$$

$$\Rightarrow \int_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$$

영역 D의 면적

$F = -yi + xj$에 Green의 정리

적용

$$\int_{\partial D} -ydx+xdy = \iint_D \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} dxdy$$

$$= 2\iint_D dxdy$$

$$\iint_D dxdy = \frac{1}{2}\int_{\partial D} -ydx+xdy$$

예) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 타원의 면적을 구해보라.

Green의 정리에 의하면

C : 타원의 경계선

$$\frac{1}{2}\int_C -ydx+xdy$$

C를 매개화 하면 $x=a\cos t, y=b\sin t, 0 \leqq t \leqq 2\pi$

$$\frac{1}{2}\int_0^{2\pi}\left[(-b\sin t)\frac{dx}{dt}+(a\cos t)\frac{dy}{dt}\right]dt$$

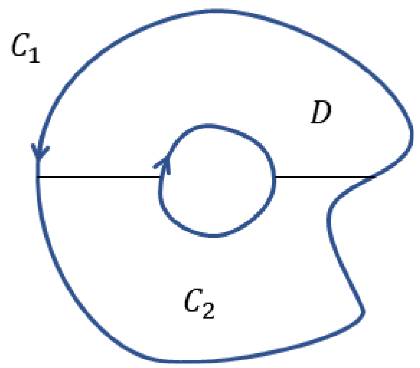
$$=\frac{1}{2}\int_0^{2\pi}[(-b\sin t)(-a\cos t)+(a\cos t)(b\cos t)]dt$$

$$=\frac{1}{2}\int_0^{2\pi}ab\sin^2 t+ab\cos^2 t dt$$

$$=\frac{1}{2}2\pi ab$$

$$=\pi ab$$

예제)



$$\partial D = C_1 \cup C_2$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$$

$$\int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$

예제)

$C=$ boundary of $\{1 \leqq x^2+y^2 \leqq \alpha\}$ with positive orientation

$$\int_C (x^3-y^3)dx + (x^3+y^3)dy$$

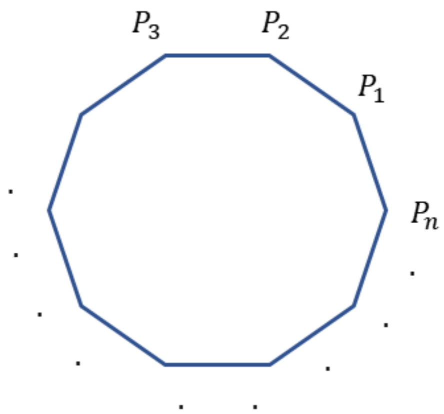
$$= \int_D \frac{\partial x^3+y^3}{\partial x} - \frac{\partial x^3-y^3}{\partial y} dxdy$$

$$= \int_D 3x^2 - (-3y^2)dxdy$$

$$= \int_0^{2\pi} \int_1^3 3r^2 rdrd\theta$$

$$= 3 \times 2\pi \int_1^3 r^3 dr$$

예제)



polygon

$$P_j = (x_j, y_j)$$

$$A = \frac{1}{2} \int_C -ydx + xdy$$

$$x(t) = (1-t)x_1 + tx_2$$

$$y(t) = (1-t)y_1 + ty_2$$

$$\int_C -ydx+xdy=\int_{t=0}^{t=1} -(ty_2+(1-t)y_1)(x_2-x_1)dt+(tx_2+(1-t)x_1)(y_2-y_1)dt$$

$$-y_1(x_2-x_1)+x_1(y_2-y_1)$$