Module Chain rule and directional derivative

1. 합성함수의 미분법 (Chain rule)

Chain rule for $z=f(x(t),y(t))$

$\frac{dz}{dt}=?$

We will show that $\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

$$\triangle z=f(x(t_0+\triangle t),y(t_0+\triangle t))-f(x(t_0),y(t_0))$$

$$\frac{x(t_0+\triangle t)-x(t_0)}{\triangle t}\approx x'(t_0)$$

$$\frac{y(t_0+\triangle t)-y(t_0)}{\triangle t}\approx y'(t_0)$$

$$\Rightarrow \begin{cases} x(t_0+\triangle t)\approx x(t_0)+x'(t_0)\triangle t \\ y(t_0+\triangle t)\approx y(t_0)+y'(t_0)\triangle t \end{cases}$$

$$\triangle z\approx f(x(t_0)+x'(t_0)\triangle t, y(t_0)+y'(t_0)\triangle t)-f(x(t_0),y(t_0))$$

$$\begin{aligned} &=f(x(t_0)+x'(t_0)\triangle t,y(t_0)+y'(t_0)\triangle t)-f(x(t_0),y(t_0)+y'(t_0)\triangle t) \\ &\quad+f(x(t_0),y(t_0)+y'(t_0)\triangle t)-f(x(t_0),y(t_0)) \end{aligned}$$

$$\frac{f(x_0+\triangle x,y_0+\triangle y)-f(x_0,y_0+\triangle y)}{\triangle x}$$

$$\approx \frac{\partial f}{\partial x}(x_0,y_0+\triangle y)$$

$$\frac{f(x_0,y_0+\triangle y)-f(x_0,y_0)}{\triangle y}\approx \frac{\partial f}{\partial y}(x_0,y_0)$$

준식 $\approx \frac{\partial f}{\partial x}(x_0,y_0+\triangle y)\frac{dx}{dt}\triangle t+\frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial t}\triangle t$

$$\Rightarrow \left.\frac{\triangle z}{\triangle t}\right|_{t=t_0}\approx \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\lim_{\triangle t\to 0}\left.\frac{\triangle z}{\triangle t}\right|_{t=t_0}=\frac{dz}{dt}=$$

$$\triangle t\to 0 \quad \Rightarrow \qquad y_0+\triangle y=y_0+y'(t_0)\triangle t\to y_0$$

$$\Rightarrow dz=f_x dx+f_y dy$$

예제)

$$w=f(x(t),y(t),z(t))$$

$$\frac{dw}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\frac{dz}{dt}$$

예제)

$$r(t)=(p_1+tv_1,p_2+tv_2,p_3+tv_3)$$

$$=P+tV$$

$$f(x,y,z)=xyz$$

$$\left.\frac{d}{dt}f(r(t))\right|_{t=0}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\frac{dz}{dt}$$

$$=yz\cdot\ v_1+xz\cdot\ v_2+xy\cdot\ v_3$$

$$=p_2p_3v_1+p_1p_3v_2+p_1p_2v_3$$

$$r(0)=P$$

$$\frac{d}{dt}f(P+tV)=\left(\frac{\partial f}{\partial x}(P),\frac{\partial f}{\partial y}(P),\frac{\partial f}{\partial z}(P)\right)\cdot\ V$$

$$=\nabla f(P)$$ gradient of f

$$P=(1,1,1)$$

$$\nabla f=(1,1,1)$$

Q: For which V , $\frac{d}{dt}f(r(t))$ has largest value and smallest values?

예제) 가스 속을 통과하는 어떤 입자가 각 지점에서 확인하는 온도는?
가스의 온도는 삼차원 공간 상의 각 위치의 함수

가스 속을 통과하는 어떤 입장의 위치 함수 $X(t) = (x_1(t), x_2(t), x_3(t))$

$f(x_1, x_2, x_3)$: 공간상의 (x, y, z) 에서 gas의 온도

$f(X(t)) = f(x_1(t), x_2(t), x_3(t))$: path상의 각 점에서 gas의 온도에 관심

Question: 입자가 진행하는 방향에 대해 온도가 얼마나 급격하게 변하는지 알 수 있는가?

=> t에 관한 f의 순간 변화율

$$t \mapsto f((X(t)) \qquad \frac{d}{dt} f(X(t))$$

예제)

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$
$$X(t) = (\cos t, \sin t, t)$$
$$f(X(t)) = (\cos t)^2 + (\sin t)^2 + t^2$$

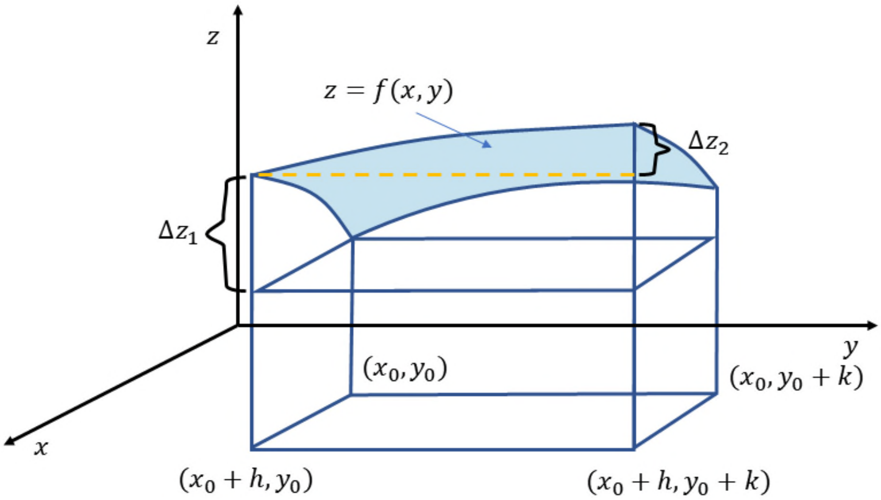
f와 X가 복잡한 경우 합성함수의 식을 얻는 다음 t에 대해 미분하는 것은효율적이지 않다.

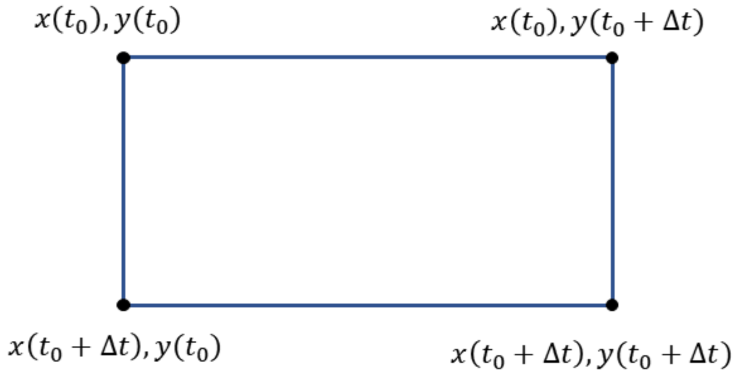
제안 => Chain rule 이용해서 미분해보기

2. Chain rule 공식에 대한 유도

변수가 2개인 경우 $f : (a,b) \times (c,d) \rightarrow R$ 일급함수, 즉 $(f, f_x, f_y$, 연속)

$$X : I \rightarrow (a,b) \times (c,d) \quad \text{미분가능}$$
$$X(t) = (x_1(t), x_2(t))$$
$$\frac{d}{dt} f(X(t)) = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t}$$





$$\triangle z = f(x(t_0 + \triangle t), y(t_0 + \triangle t)) - f(x(t_0), y(t_0))$$

$$\triangle z_1 = f(x(t_0 + \triangle t), y(t_0)) - f(x(t_0), y(t_0))$$

$$\triangle z_2 = f(x(t_0 + \triangle t), y(t_0 + \triangle t)) - f(x(t_0 + \triangle t), y(t_0))$$

$$\triangle z = \triangle z_1 + \triangle z_2$$

$$\triangle z_1 = f'(x(t_0 + \xi \triangle t))x(t_0 + \triangle t) - x(t_0)$$

$$\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$$

$$\triangle z_1 = f(x_0 + \triangle x, y_0) - f(x_0, y_0)$$

$$\triangle z_2 = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0 + \triangle x, y_0)$$

Apply mean value theorem to $x \mapsto f(x, y_0)$

$$[x_0, x_0 + \triangle x]$$

$$\triangle z_1 = f(x_0 + \triangle x, y_0) - f(x_0, y_0)$$

$$= \frac{\partial f}{\partial x}(x_0 + \xi \triangle x, y_0) \triangle x$$

Apply mean value theorem to $y \mapsto f(x_0 + \Delta x, y)$

$$[y_0, y_0 + \Delta y]$$

$$\Delta z_2 = \frac{\partial f}{\partial y}(x_0 + \Delta x, y_0 + \eta \Delta y)\Delta y$$

$$\Delta z_1 = \left(\frac{\partial f}{\partial x}(x_0, y_0) + \epsilon_1\right)\Delta x$$

$$\Delta z_2 = \left(\frac{\partial f}{\partial y}(x_0, y_0) + \epsilon_2\right)\Delta y$$

$$\Delta z = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

when $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt}$$

▪ 변수가 3개인 경우

$$w = f(x, y, z) \qquad x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

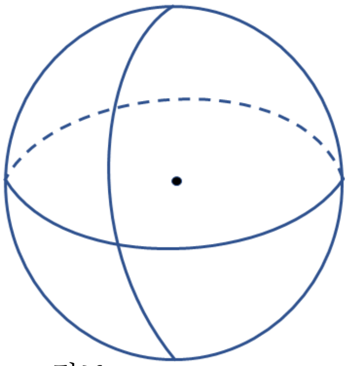
$$\frac{dw}{dt} = \frac{d}{dt} f(x(t), y(t), z(t))$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

▪ 변수가 3개인 함수와 변수가 2개인 함수의 합성 $x = x(s,t), y = y(s,t), z = z(s,t)$

예를 들면 구면위에서 정의된 함수를 다룰 경우. 구면을 삼차원 공간 상에 있다고 하자. 삼차원 공간 상에 정의 된 함수를 구면에 제한 시켜 다루는 경우. 구면은 2차원 곡면이므로 두 개의 매개변수로 설명할 수 있다. 예를 들면 지구 상의 위치를 경도와 위도 두 변수만으로 위치를 확정할 수 있는 것과 같다.
구면 좌표계에서 반경을 1이라 놓으면 구면에 대한 두 변수의 매개변수식을 얻을 수 있다.

$x = x(s,t), y = y(s,t), z = z(s,t)$로 표현하자.



$s=$ 경도

$t=$ 위도

$$\frac{\partial}{\partial s} f(x(s,t), y(s,t), z(s,t))$$

$$\frac{\partial}{\partial t} f \cdot \quad X(s,t)$$

$$\frac{\partial f \cdot \quad X}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

예제)

$$f(x_1, x_2) = x_1^3 + 2x_1 x_2$$

$$x_1(r,\theta) = r\cos\theta$$

$$x_2(r,\theta) = r\sin\theta$$

$$f(x_1(r,\theta), x_2(r,\theta)) = f(X(r,\theta))$$

$$\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta} = ?$$

Sol)

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial r}$$

$$= (3x_1^2 + 2x_2)\cos\theta + 2x_1\sin\theta$$

$$= (3r^2\cos^2\theta + 2r\sin\theta)\cos\theta + 2r\cos\theta\sin\theta$$

$$= 3r^2\cos^3\theta + 4r\cos\theta\sin\theta$$

$$r=\sqrt{x_1^2+x_2^2} \qquad \cos\theta=\frac{x_1}{\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial f}{\partial r}=\frac{(3x_1^2+2x_2)x_1+2x_1x_2}{\sqrt{x_1^2+x_2^2}}=\frac{3x_1^3+4x_1x_2}{\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \theta}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \theta}$$

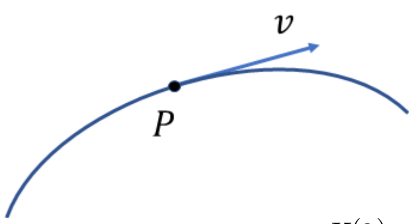
$$=\frac{\partial f}{\partial x_1}(-r\sin\theta)+\frac{\partial f}{\partial x_2}r\cos\theta$$

$$=-3r^3\cos^2\theta\sin\theta-2r^2\sin^2\theta+2r^2\cos^2\theta$$

$$=-3r^3\cos^2\theta\sin\theta+2r^2\cos^2\theta$$

3. Directional derivative 방향 편미분

$D_{\vec{v}}f(P)$: f의 P 에서의 $\vec{v}$ 방향의 방향 편미분



$$X(0)=P$$

$$X'(0)=\vec{v}$$

$$\left.\frac{d}{dt}f(X(t))\right|_{t=0}=\left.\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{dx_j}{dt}\right|_{t=0}$$

(정의) $\nabla f(P)=\left(\frac{\partial f}{\partial x_1}(P),\cdots,\frac{\partial f}{\partial x_n}(P)\right)$: gradient of f at P

$$X'(0)=\left(\frac{dx_1}{dt}(0),\cdots,\frac{dx_n}{dt}(0)\right)=\vec{v}$$

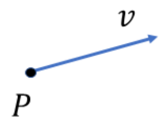
$$\left.\frac{d}{dt}f(X(t))\right|_{t=0}=\nabla f(P)\cdot\ \vec{v}$$

$$D_V f(P)=\nabla f(P)\cdot\ V \quad \text{(핵심 공식)}$$

예제)

$$f(x,y,z)=x^3-xy^2-z$$

$$P=(1,1,0) \qquad V=(2,-3,6)$$



$$D_V f(P)=\nabla f(P)\cdot V$$

$$\nabla f(X)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

$$\frac{\partial f}{\partial x}=3x^2-y^2$$

$$\frac{\partial f}{\partial y}=-2xy$$

$$\frac{\partial f}{\partial z}=-1$$

$$\nabla f(P)=\left(3(1)^2-(1),-2(1)(1),-1\right)$$

$$=(2,-2,-1)$$

$$D_V f(P)=(2,-2,-1)\cdot (2,-3,6)$$

$$=(2)(2)+(-2)(-3)+(-1)(6)$$

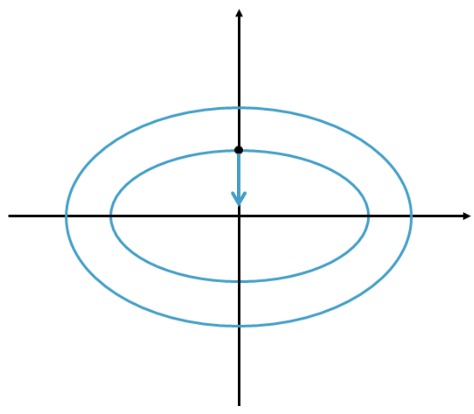
$$=4+6-6=4$$

응용 문제) 가열된 철판의 각 점에서의 온도가 $f(x,y)=273\mathrm{e}^{-2x^2-3y^2}$ 로 주어져 있다. 원점에서 온도가 가장 높고 원점에서 멀어질수록 온도는 내려간다.

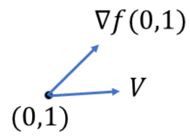
$(0,1)$에서 온도가 가장 빨리 증가하는 방향은 어디인가?

Sol)

$$f=c \quad \Leftrightarrow \quad 2x^2+3y^2=c$$



$$\text{Find a vector } V \quad s.t \quad D_V f(0,1) \text{ has largest value}$$
$$= \nabla f(0,1) \cdot V$$
$$= |\nabla f(0,1)||V| \cos\theta$$
$$= |\nabla f(0,1)| \cos\theta$$



$\theta = 0$ 일 때 가장 큼

$$\text{Take } V = \frac{\nabla f(0,1)}{|\nabla f(0,1)|}$$

$$\nabla f = 273\mathrm{e}^{-2\mathrm{x}^2 - 3\mathrm{y}^2}\left(\frac{\partial}{\partial \mathrm{x}}(-2\mathrm{x}^2 - 3\mathrm{y}^2), \frac{\partial}{\partial \mathrm{y}}(-2\mathrm{x}^2 - 3\mathrm{y}^2)\right)$$
$$(-4x, -6y)$$
$$(-4(0), -6(1)) = (0, -6)$$
$$V = (0, -1)$$

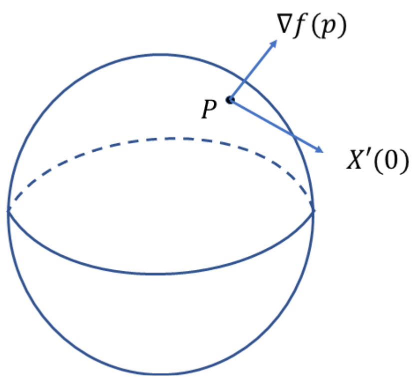
$\Rightarrow \nabla f(p)$는 p에서 함수 값이 가장 크게 변하는 방향

$\Rightarrow \theta = \frac{2}{2} \qquad \frac{2}{2} V \perp \nabla f(P) \quad D_V f(P) = 0$

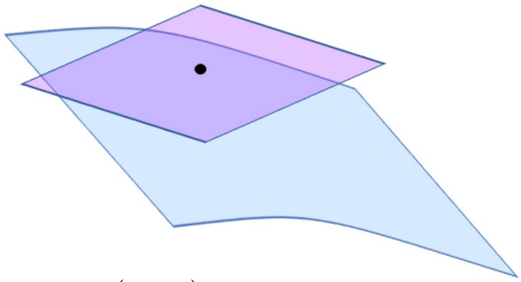
함숫값이 변하지 않는 방향

4. f의 gradient의 기하학적 의미

$$f(x(t), y(t), z(t)) = c$$
$$\frac{d}{dt} f \cdot X(t) = 0$$
$$\sum_j \frac{\partial f}{\partial x_j} \frac{dx_j}{dt} = 0 \qquad \nabla f \cdot X' = 0$$



$\nabla f(p)$ is normal vector



응용) Q: $F(x,y,z)=C$ 접평면 어떻게 정의?

$F(P)=c$

P에서의 접선 벡터들을 포함하는 Space가 접평면이다. 접평면은 법선벡터에 의해 결정된다.
위에서 살펴본 것처럼 법선벡터는 F의 gradient이다.