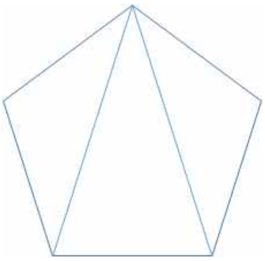
Module Integration: introduction

- Area problem
- Definite integral
- Right-hand sum and left-hand sum

Question) How can we define and compute the area of the region bounded by closed curve?

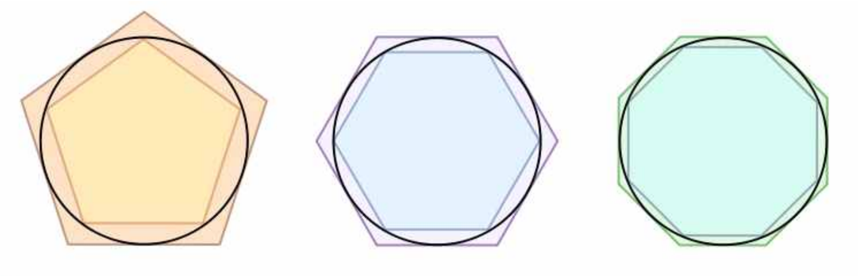
Area of polygonal region => Divide it into triangles



Area of circle

Ancient Egyptian => approximate the circle with a square

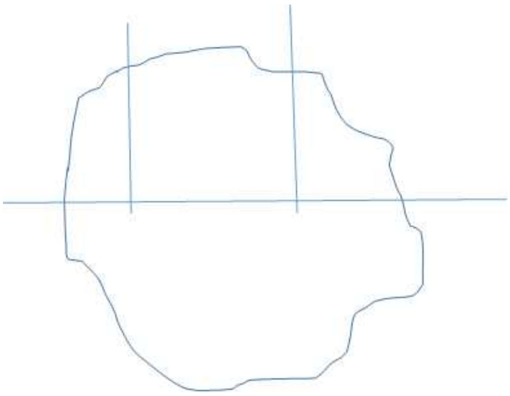
Archimedes => method of exhaustion



1. The area problem

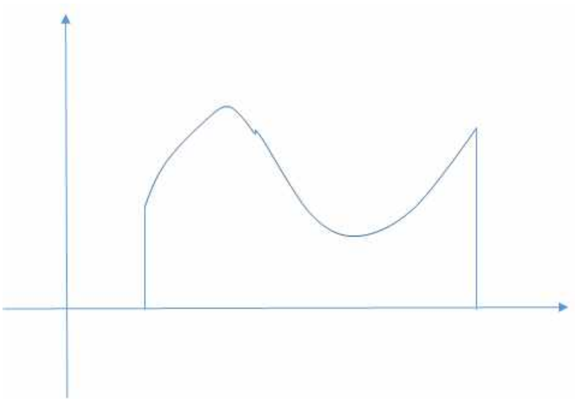
The region S is bounded by a closed curve. Define and compute the area of the region.

=> Reduce the problem into area problem for the region bounded by three straight lines and a curve (=graph of a function)



Basic setting

The graph of a continuous function f (f >0), the vertical lines x=a and x=b, and the x-axis. The region S determined by these lines and curves.



Idea: Slice the region in vertical direction => vertical strips (approximately rectangles)

Example The area under the curve $y=x^2$ from x=0 and x=1

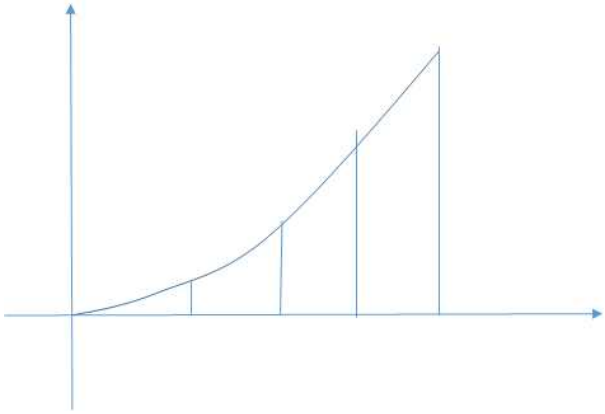
1) Divide S into four strips by vertical lines x=1/4, x=1/2, x=3/4.

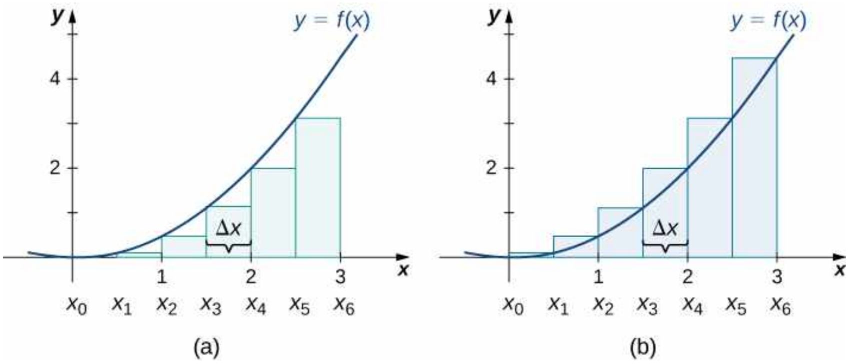
2) Approximate each strip by a rectangle whose base equal to the strip and whose height is the right edge of the strip (right edge rule)

-How can we determine the length of the right edge of the strip?

3) Sum of the areas of these approximating rectangles

=> $R_4 = (1/4)(1/4)^2 + (1/4)(1/2)^2 + (1/4)(3/4)^2 + (1/4)(1)^2 = 15/32$ (right-hand sum)





4) Repeat the step 2 with rectangle whose height is the left edge of the strip (left edge rule).

5) Repeat the step 3

=> $L_4 = (1/4)(0)^2 + (1/4)(1/4)^2 + (1/4)(1/2)^2 + (1/4)(3/4)^2 = 7/32$ (Left-hand sum)

=> 7/32 =L_4 < A < R_4=15/32

Error is less than R-4- L_4 =(15-7)/32 =8/32 =1/4

6) (Improve approximation) Repeat the whole procedure with a larger number of strips

=> Take 8 strips

coordinates for cutting points : x = k/8, k=0, 1, 2, .., 8

$R_8 = (1/8)(1/8)^2 + \cdots + (1/8)(8/8)^2$ (right-hand sum with 8 rectangles)

$L_8 = (1/8)(0)^2 + \cdots + (1/8)(7/8)^2$

=> L_8 < A <R_8 and Error= (R_8 - L_8)< (R_4 - L_4)

R_n = sum of areas of n approximating rectangles with right edges

L_n = sum of areas of n approximating rectangles with left edges

Claim $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$ => area of the region

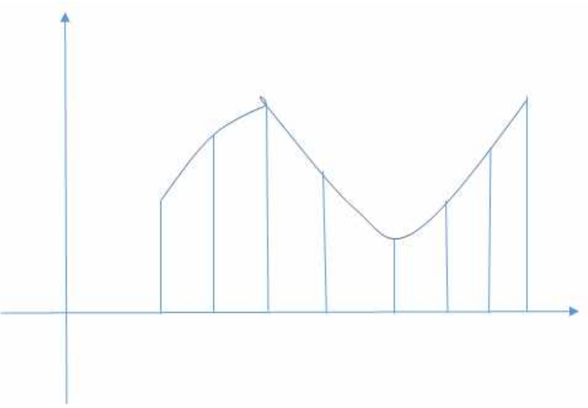
width of each strip = 1/n

$$R_n = \frac{1}{n}(1/n)^2 + \frac{1}{n}(2/n)^2 + \cdots + \frac{1}{n}(n/n)^2$$

$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots n^2) = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \rightarrow \frac{2}{6}$$

Exercise) Compute L_n and show that its limit is 1/3 as n -> infinity.

General Case



x is between x=a and x=b, graph of $y = f(x)$

width of each strip = $\frac{(b-a)}{n} = \Delta x$

Divide the interval [a, b] into n sub-intervals, width of each interval is $\frac{(b-a)}{n} = \Delta x$

=> $x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{n-1} < x_n$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$x_1 = x_0 + \Delta x = a + \frac{(b-a)}{n}$$
$$x_2 = x_0 + 2\Delta x = a + 2\frac{(b-a)}{n}$$

$$x_i = x_0 + i\Delta x = a + i\frac{(b-a)}{n}$$

$$R_n = \sum_{i=1}^{n} f(a + i\frac{(b-a)}{n})\Delta x$$

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Sigma notation (for those who did not learn at high school)

Please see Appendix E, Stewart for detail.

$a_1, a_{2,} \ldots, a_m$: real numbers

$$\sum_{i=1}^{m} a_i = a_1 + a_2 + \cdots + a_m$$

summation in which the letter i

(index of summation) takes on consecutive integer values beginning with 1 and ending with n.

=> a_i is a formula for the ith term

(Any letter can be used to denote the index but letter i, j, k are customary.

Example)

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

$$\sum_{i=1}^{n-1} a_i \quad (a_1 = 2^3,\ a_2 = 3^3,\ \ \ldots\ \ , a_i = (1+i)^3 \quad)$$

=> $\sum_{i=1}^{n-1} (1+i)^3$

Properties

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

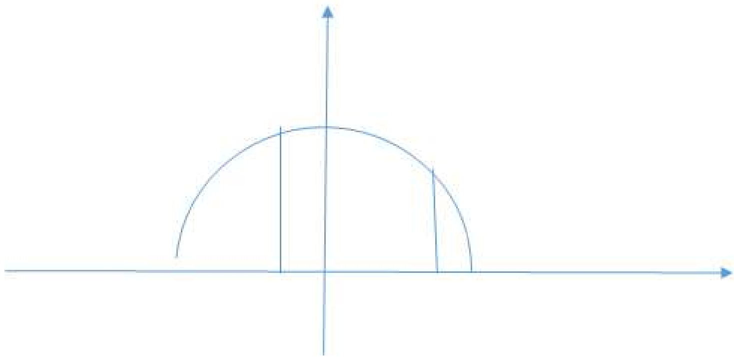
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Example)

$$\sum_{k=1}^{n} 2k^2 - k = 2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k = 2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

Back to Area Problem

Example Find the area of the region below $y = 25 - x^2$ over $-1 \le x \le 4$ using 5 rectangles and n rectangles



a=-1, b=4

$\Delta x = (b-a)/n = 5/5 = 1$

$x_0 = -1,\ x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = 4$

$$R_5 = \sum_{i=1}^{5} f(x_i)\Delta x = \sum_{i=1}^{5} f(-1+i)(1) = \sum_{i=1}^{5} (25 - (-1+i)^2)$$

$$= 25 + (25 - 1^2) + (25 - 2^2) + (25 - 3^2) + (25 - 4^2)$$

$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\Delta x = \sum_{i=1}^{5} (25 - (x_{i-1})^2) = \sum_{i=1}^{5} (25 - (i-2)^2)$$

For n strips, x_i = -1+ i Delta a = -1 + i (5/n)

$$R_n = \sum_{i=1}^{n} (25 - (-1 + \frac{5i}{n})^2)\frac{5}{n} = \sum_{i=1}^{n} \frac{125}{n} - \sum_{i=1}^{n} (1 - 10\frac{i}{n} + \frac{25i^2}{n^2})\frac{5}{n}$$

$$= 120 + \frac{50}{n^2}\sum_{i=1}^{n} i - \frac{125}{n^3}\sum_{i=1}^{n} i^2$$

$$= 120 + \frac{50}{n^2}\frac{n(n+1)}{2} - \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6}$$

-> 120+25 – 125/3 = 310/3 (actual area of the region)

Exercise) Find L_n and show that its limit = 310/3