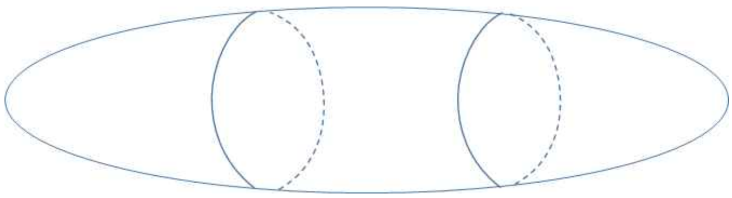
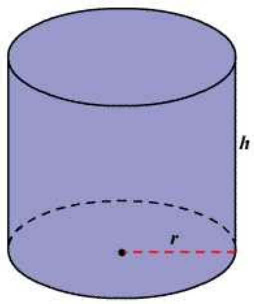
Module Volume of solid

1. Volume of Solid (Stewart section 5.2)

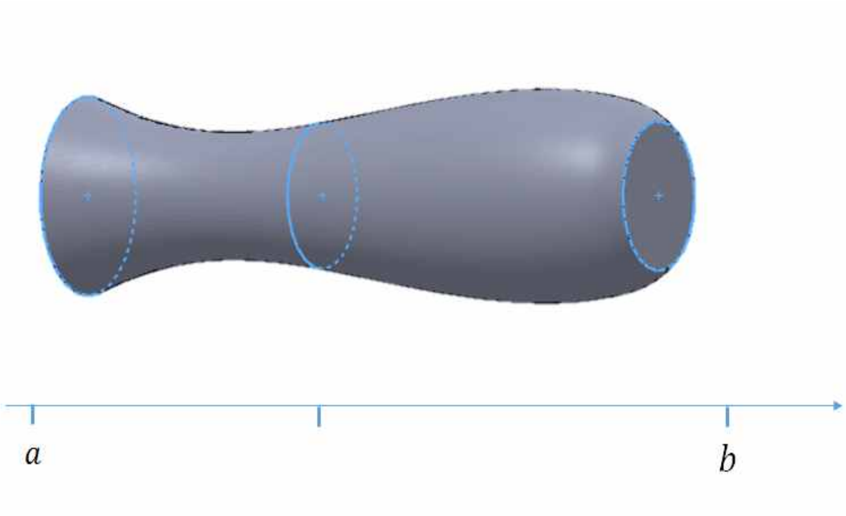


We want to find out the volume of solid using integration.
Recall that the volume of <u>circular cylinder</u>. The circular cylinder is simple type of solid in the sense that each section is identical circle. Thus its volume is simply base area × height where the base is circle. The circular cylinder is prototype of the solids which we will consider.

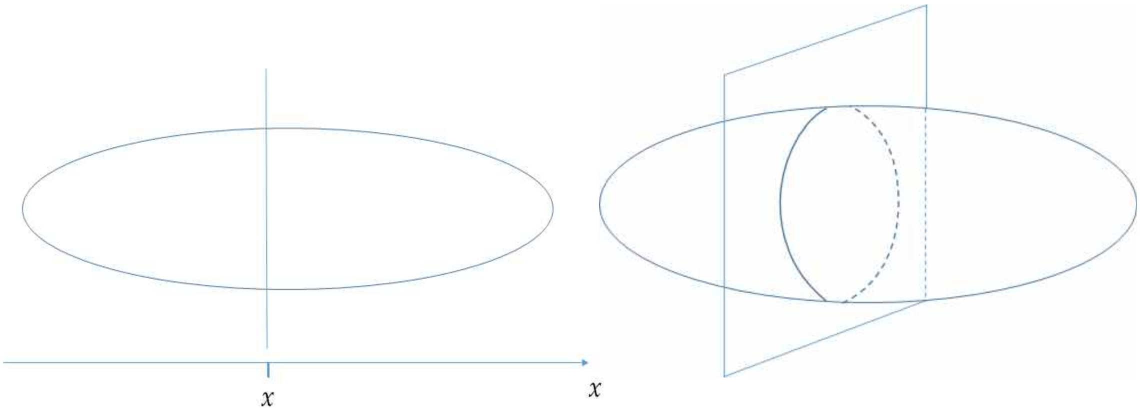
Typical solid we consider is like a rugby ball.
<u>Key idea of finding its volume : cut the solid into pieces and approximate each piece by cylinder.</u>



Definition
The solid S is placed along the x-axis. The planar region obtained by intersecting S with a plane perpendicular to x-axis and passing through x is called **<u>cross-section</u>** of S at x.



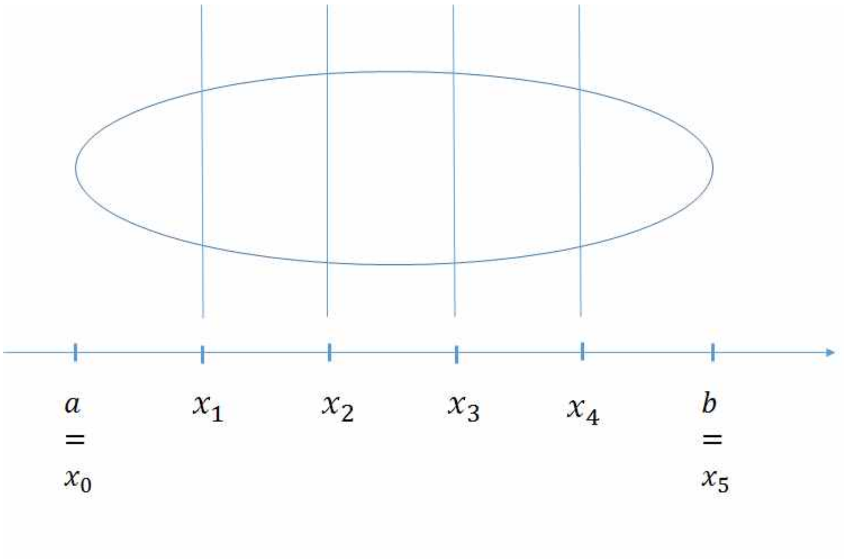
For given solid S, we introduce a function $A(x)$, which is the area of cross-section of S at x.



*How to find the volume of solid?

We first divide S into n slabs of equal width $\triangle x$.

The picture below shows the case of $n = 5$.



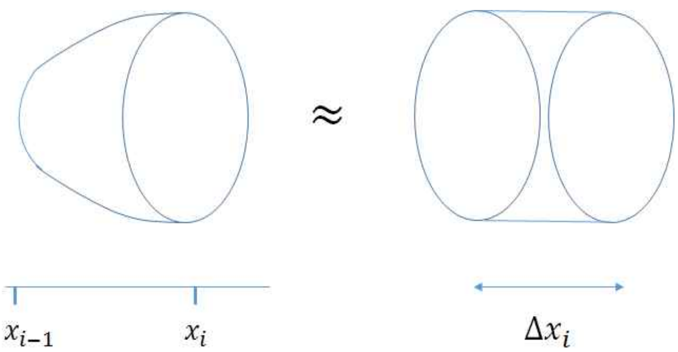
Approximate the volume of ith slab with the volume of cylinder.

The volume of approximating cylinder is base × height where

base area = area of cross section at $x_i = A(x_i)$

height = length of sub-internal = Δx.

=> the volume of approximating cylinder is $A(x_i)\Delta x$.



B summing up all the volumes of the approximating cylinders, we have an approximation of the volume V of S:

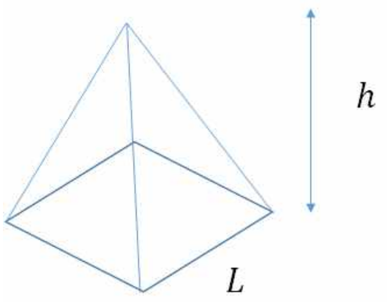
$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x.$$

Note that the sum is right-hand sum of the function $A(x)$ over $a \le x \le b$.

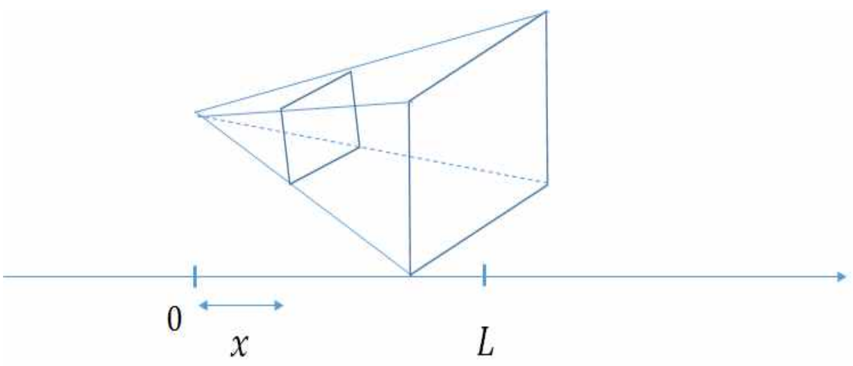
=>If we divide into more slabs taking smaller Δx, then the sum of volumes of approximating cylinders converges to V :

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx$$

Example) Find the volume of a pyramid whose base is square with side L and whose height is h.



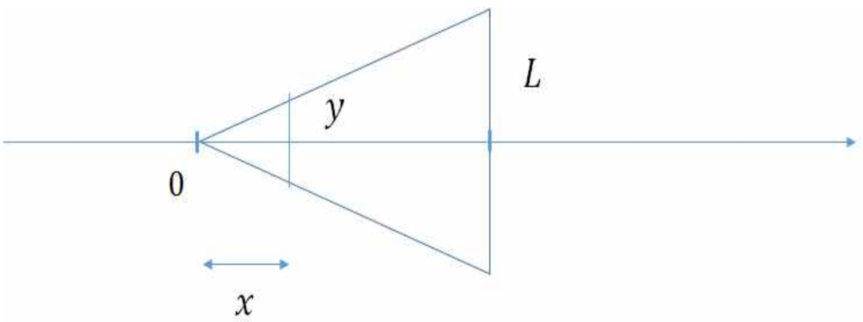
We place the pyramid along the x-axis such that the x-axis coincides with the central vertical axis of the pyramid. Then its cross section is square.



(pyramid is placed over interval between 0 and h)

Since the side view of the pyramid is a triangle, we can find out the side length of the cross section at x using similarity of triangles:

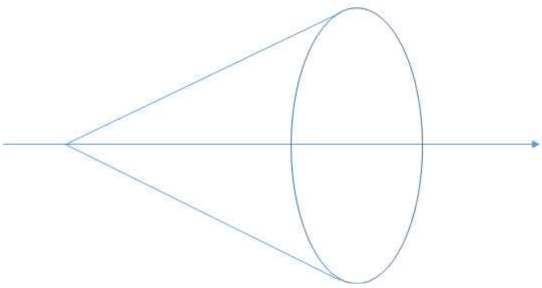
$$\frac{y}{x} = \frac{L}{h} \qquad y = \frac{L}{h}x$$

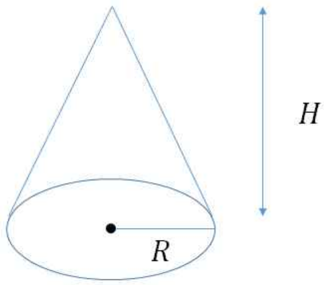


Then area of the cross section at x is the area of square of side $(L/h)x$.
=> A(x)=((L/h)x)^2. Thus

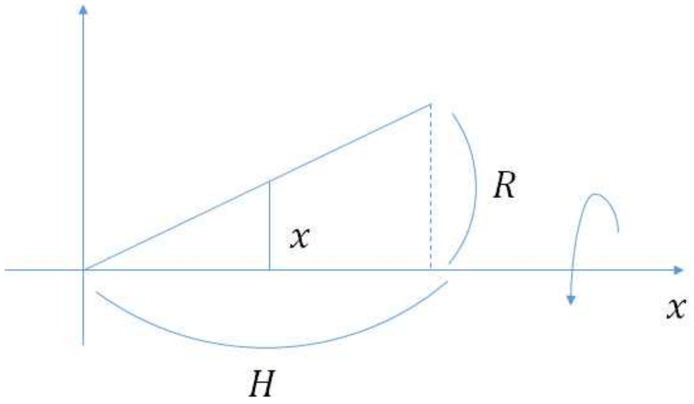
$$V = \int_0^h A(x)\,dx = \int_0^h (\frac{L}{h}x)^2\,dx = \frac{1}{3}L^2 h$$

Example) Find the volume of right circular cone.





We can consider the right-circular cone as a <u>solid of revolution.</u>



The solid of revolution is obtained by rotating the region about x-axis. We rotate the right triangle one of whose side is on the x-axis.

Then the cross section is circle.

Thus the area of the cross section at x is A(x)=$\pi r(x)^2$ where $r = r(x)$ is the height of the line. Using similarity, we have

$$\frac{R}{H} = \frac{r}{x} \qquad r = \frac{R}{H}x$$

Thus the area of the cross section at x is

$$A(x) = \pi(\frac{R}{H})^2 x^2$$

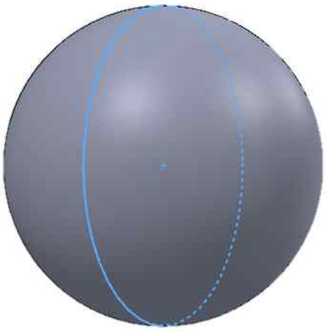
and the volume of the solid is

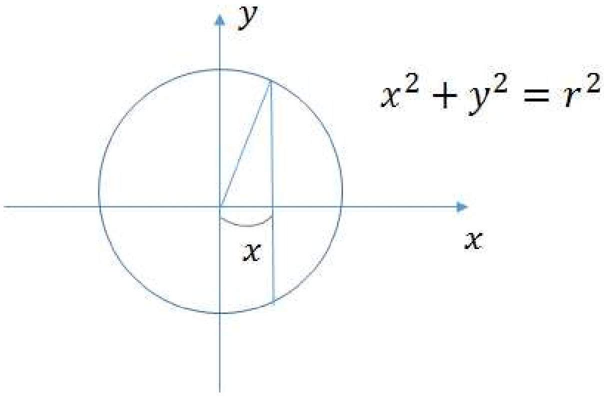
$$V = \int_0^H A(x)\,dx = \int_0^H \pi(\frac{R}{H})^2 x^2\,dx$$

$$= \pi(\frac{R}{H})^2 \frac{1}{3} x^3 \Big|_{x=0}^{x=H}$$

$$= \pi(R/H)^2 \frac{1}{3} H^3 = \frac{\pi}{3} R^2 H$$

Example) Volume of sphere of radius r





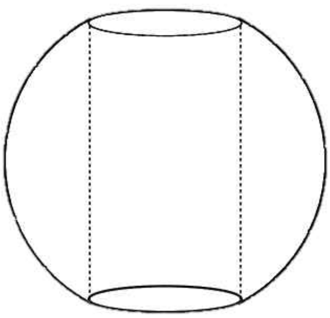
Sphere can be considered as the solid of revolution. It is obtained by rotating the circle about x-axis. Then the area of the cross section is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Thus the volume is

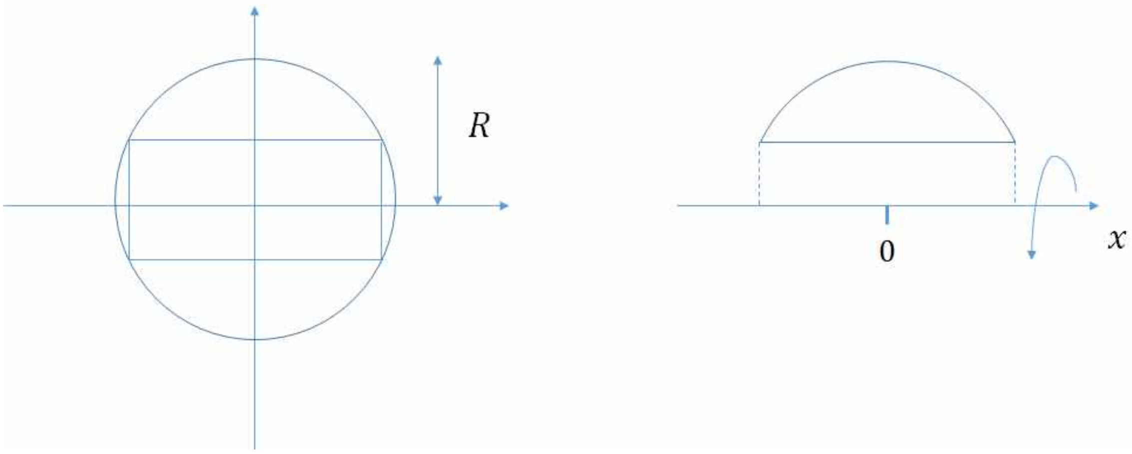
$$\begin{aligned} V = \int_{-r}^{r} A(x)\,dx &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\ &= \pi\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{x=-r}^{x=r} \\ &= \pi\left(2r^3 - \frac{4}{3}r^3\right) \\ &= 2\pi\frac{2}{3}r^3 = \frac{4\pi}{3}r^3 \end{aligned}$$

Example)



A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

The solid described can be realized as an solid of revolution by rotating the region between the circle and horizontal line.



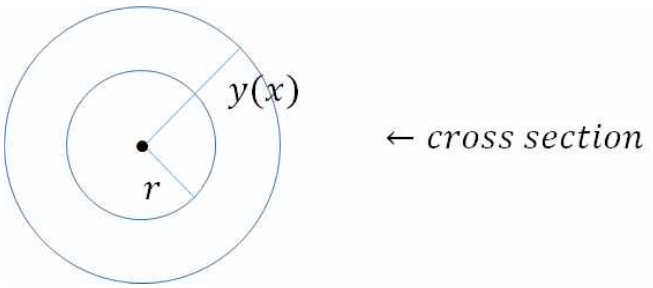
The height of the horizontal line is determined by the radius of the hole.
Note that some parts of both ends of the original sphere are removed. Thus we need to determine the interval on which the region is placed.

We solve

$$\begin{cases} x^2 + y^2 = R^2 \\ y = r \end{cases}$$

then we get $x = \pm\sqrt{R^2 - r^2}$.

Note that cross section is ring shaped region.



Thus area of the cross section is

$$\begin{aligned} A(x) &= \pi y^2 - \pi r^2 \\ &= \pi(R^2 - x^2) - \pi r^2 \\ &= \pi(R^2 - r^2 - x^2) \end{aligned}$$

Now we obtain the volume of the solid as

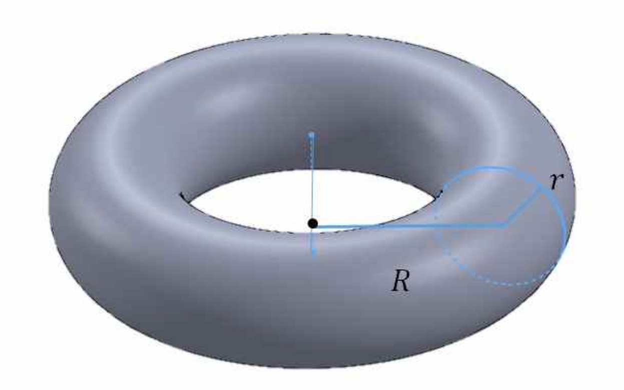
$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\,(R^2 - r^2 - x^2)\,dx$$

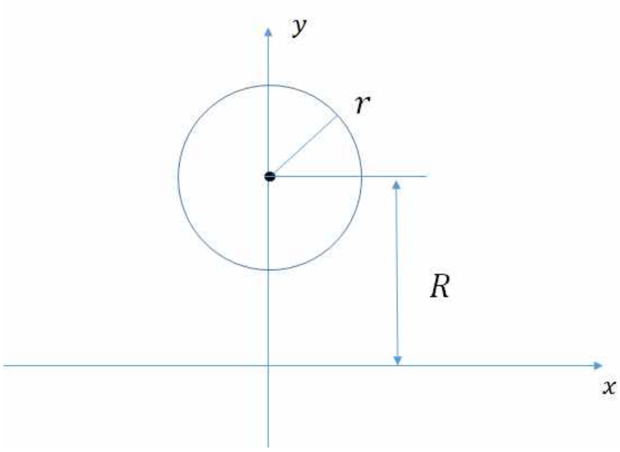
$$= \pi\,[\,(R^2-r^2)\,x - \frac{1}{3}x^3]\,\Big|_{x=-\sqrt{R^2-r^2}}^{x=\sqrt{R^2-r^2}}$$

$$= \pi\left[(R^2-r^2)\sqrt{R^2-r^2} - \frac{2}{3}\sqrt{R^2-r^2}\,(R^2-r^2)\right]$$

$$= \frac{4\pi}{3}\,(R^2-r^2)\sqrt{R^2-r^2}$$

Example) Find the volume of torus (ring donut).





We can consider the torus as solid of revolution.

We rotate the circle of radius r whose center apart from x-axis by R about x-axis.

=>Then the cross section is ring-shaped region.

Decide the radius of outer circle and the radius of inner circle.

The height of upper part of the circle $x^2+(y-R)^2=r^2$ => the radius of the outer circle => $R+\sqrt{r^2-x^2}$

The height of lower part of the circle $x^2+(y-R)^2=r^2$ => the radius of the inner circle => $R-\sqrt{r^2-x^2}$

$$\begin{aligned} V &= \int_{r}^{r} A(x)dx \\ &= \int_{-r}^{r} [\pi(R+\sqrt{r^2-x^2})^2 - \pi(R-\sqrt{r^2-x^2})^2]\,dx \\ &= 4\pi R\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = 4\pi R\frac{\pi r^2}{2} = 2\pi^2 Rr^2 \end{aligned}$$

the last integral is the area of semicircle with radius r.