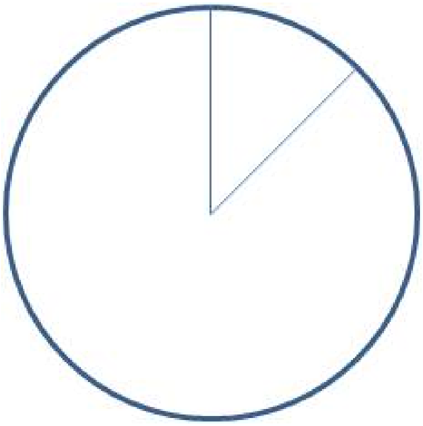
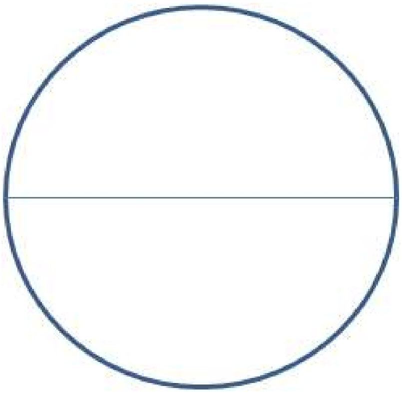
Module Trigonometric functions

1. Trigonometry

(1) Angle





1 radian: angle for circular sector whose radius=r and arc length =r

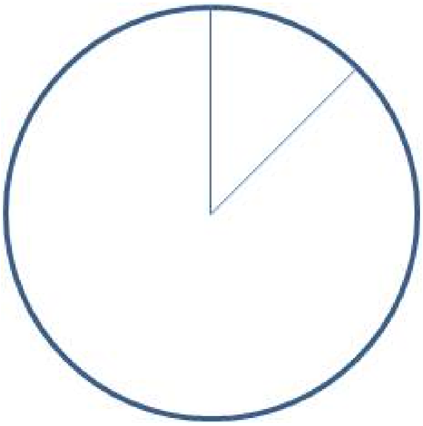
=> angle for semi circle : π radian (arc length =πr)

=> π rad = 180 DEG

=> 1 rad = 180/π DEG

=> 1 DEG = π/180 rad

DEG	30	45	60	90	120
rad	30/180 π =π/6	45/180 π =π/4	60/180 π =π/3	90/180 π =π/2	120/180 π =2π/3

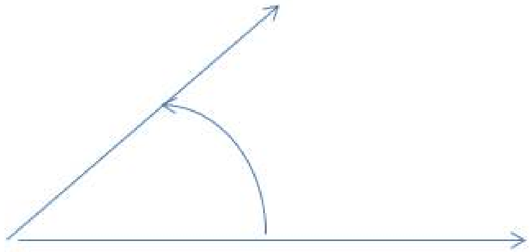


radius = r, angle=θ, =>To find arc length L, $\frac{\theta}{2\pi} = \frac{L}{2\pi r} \Rightarrow L = r\theta$

Example) circular sector whose radius=3 and angle = 30 DEG

Arc length => L= $3 \times \pi/6 = \pi/2$ $(30^{\circ} = \pi/6)$

(2) Standard position of an angle

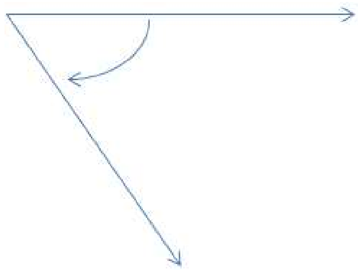


From initial side (horizontal line directed to the right) to terminal side, rotated by angle θ

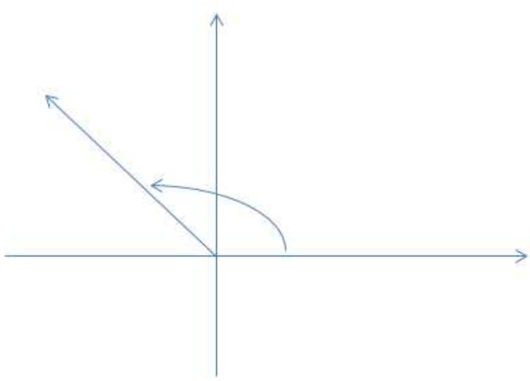
Orientation of angle:

positive angle => counter clockwise rotation of initial side

negative angle => clockwise rotation of initial side



Example)



If the line passes through (-1,1), then it makes an angle 90 DEG + 45 DEG = 135 DEG =>

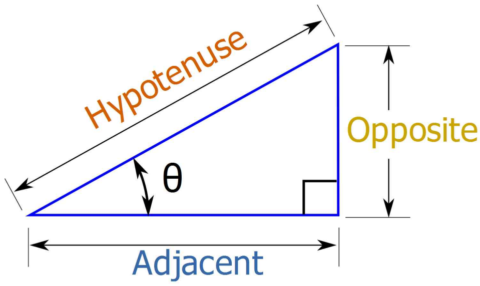
understood as counter clockwise rotated => + 135 DEG = $\pi/2+\pi/4=\frac{3}{4}\pi$

understood as clock wise rotated => $-\pi-\pi/4=-5\pi/4$

=> representation of angle is not unique $\theta+2n\pi$ (n : integers)

=> $-\frac{5}{4}\pi=\frac{3}{4}\pi-2\pi$

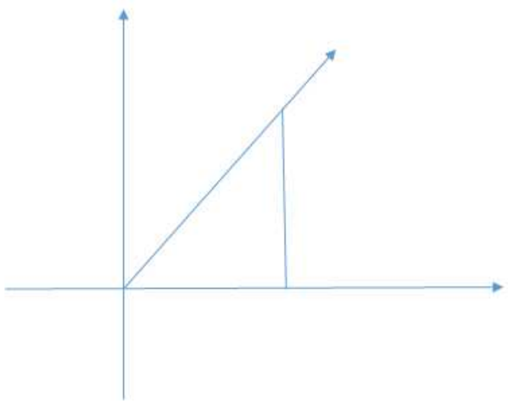
(3) Trigonometric functions



$\sin\theta=\frac{opp}{hyp}$, $\cos\theta=\frac{adj}{hyp}$, $\tan\theta=\frac{opp}{adj}$

$\csc\theta=\frac{1}{\sin\theta}$, $\sec\theta=\frac{1}{\cos\theta}$, $\cot\theta=\frac{1}{\tan\theta}$

What if angle is obtuse or negative?

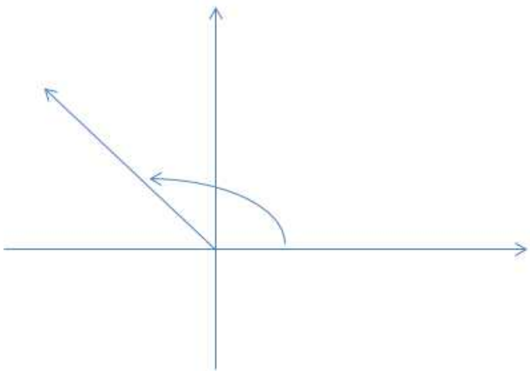


Point P on the terminal side => P (x, y), angle = θ

r= distance from the origin to the point P (=hypotenuse of the right triangle)

=> $\cos\theta=x/r$, $\sin\theta=y/r$, $\tan\theta=y/x$

Example)

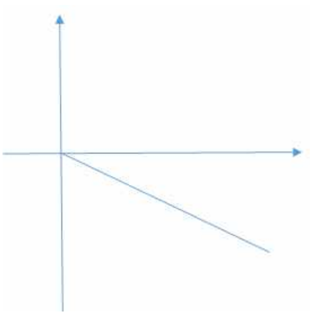


angle= $\frac{3\pi}{4}$, Value of trigonometric functions=?

A point P on the terminal side => P=$(-1,1)$ => x=-1, y=1, $r=\sqrt{2}$

$\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}$, $\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$, $\tan\frac{3\pi}{4}=-\frac{1}{1}=-1$

Example)



Angle =$-\frac{\pi}{6}$ => point P=$(\sqrt{3},-1)$, distance from the origin to P = 2

$\cos(-\frac{\pi}{6})=\frac{x}{r}=\frac{\sqrt{3}}{2}$, $\sin(-\frac{\pi}{6})=\frac{y}{r}=-\frac{1}{2}$, $\tan(-\frac{\pi}{6})=-\frac{1}{\sqrt{3}}$

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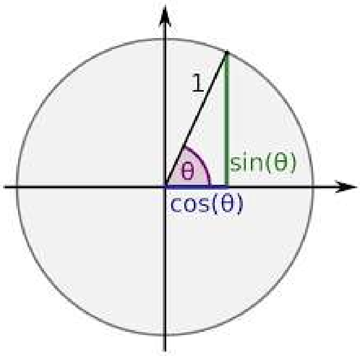
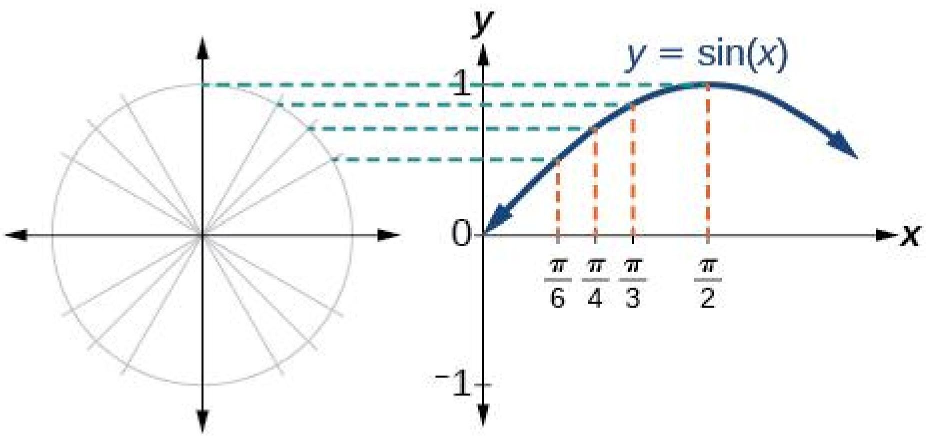
(4) Graph of sine and cosine

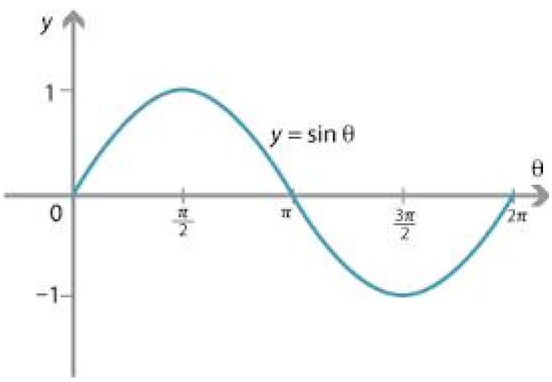


Take a unit circle, point moves along the unit circle counter clockwise.

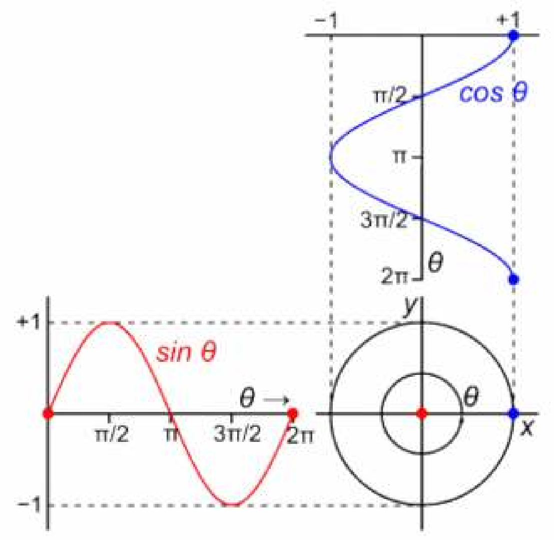
=> $x = \cos\theta$, $y = \sin\theta$ for given point P (x, y)

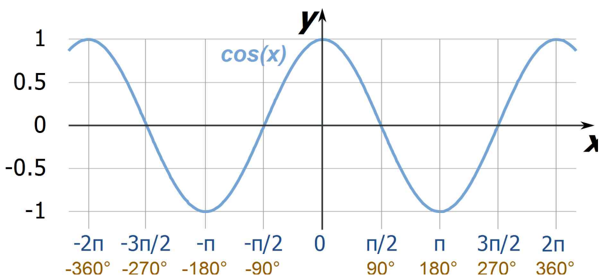
Graph of sine =>





Graph of cosine





=> Sine and cosine function are periodic

$\sin(\theta+2\pi)=\sin\theta, \quad \cos(\theta+2\pi)=\cos\theta$

(5) Identities of trigonometric functions

(a) $\cos^2\theta+\sin^2\theta=1 \quad (x^2+y^2=r^2 \Rightarrow \frac{x^2}{r^2}+\frac{y^2}{r^2}=1)$ (P=(x, y))

(b) $\cos(-\theta)=\cos\theta, \ \sin(-\theta)=-\sin\theta$ (even function / odd function)

(c) $\cos(\pi-\theta)=-\cos\theta, \ \sin(\pi-\theta)=\sin\theta$ (for angle symmetric wrt y-axis)

$\cos(\pi+\theta)=-\cos\theta, \ \sin(\pi+\theta)=-\sin\theta$ (for angle symmetric wrt the origin)

(because (x, y) -> (-x, -y))

(d) Addition formula (IMPORTANT!!)

Question) $\sin(45^\circ) = \frac{1}{\sqrt{2}}$, $\sin(30^\circ) = \frac{1}{2}$

$\sin(75^\circ) = ?$, $\sin(45^\circ + 30^\circ) = \sin(45^\circ) + \sin(30^\circ)$? NO!!

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

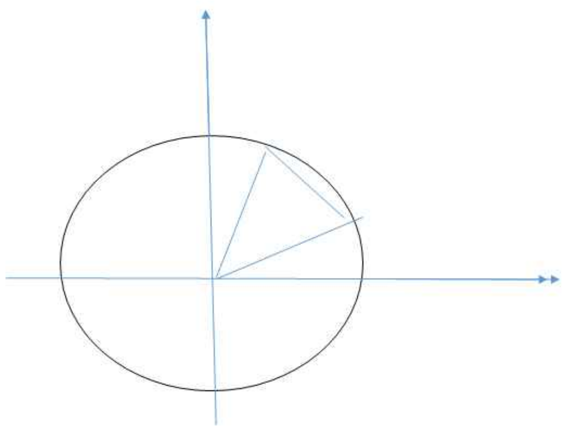
$$\cos(\alpha+\beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$$

=> sin (75)=sin (45+30) = sin 45 cos3 0 + cos 45 sin 30 (<= correct formula)
sin (15) = sin (45-30) = sin 45 cos 30 -cos 45 sin 30

Idea of proof of the cosine addition formula.
Prove: cosine subtraction formula (=> can derive other cases)
$\cos(\alpha-\beta) = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$

Consider two points P=$(\cos\alpha, \sin\alpha)$ and Q=$(\cos\beta, \sin\beta)$ on unit circle, the x-y plane. (We may assume that $\alpha > \beta > 0$)
=>Express the distance between P and Q in two different ways.
(i) Euclidean distance squared $\overline{PQ}^2$ = $(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2$.
(ii) Consider triangle whose vertices are P, Q, and the origin. Use the second cosine law to get the side length PQ. Then one has

$$PQ^2 = 1 + 1 - 2\cos(\alpha - \beta)$$



=> (i) = (ii)

$$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = PQ^2 = 1 + 1 - 2\cos(\alpha-\beta).$$

=> cosine addition formula

To obtain sine addition formula, note that $\sin(\alpha+\beta)=-\cos(\alpha+\beta+\pi/2)$ and apply cosine addition formula for the angles α and $\beta+\pi/2$.

(Use $\cos(\theta+\pi/2)=-\sin\theta$. Please verify why it is true)