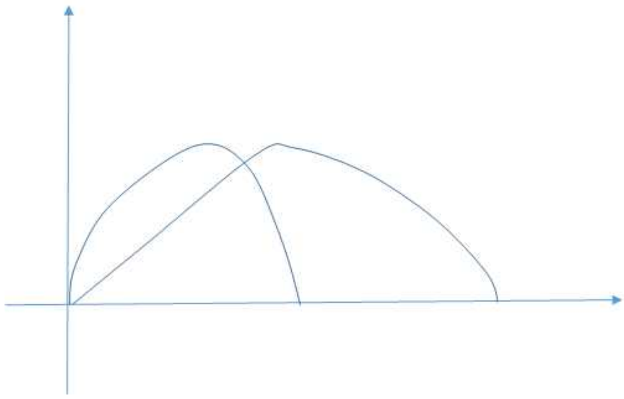
Module Substitution rule

1. Substitution rule

The number of types of functions we can integrate are limited as long as we use basic rules we introduced previous sections. In this section, a quite useful technique of integration is introduced to cover various type of functions.

Example) Evaluate $\int_0^{\pi/2} \sin 2x\, dx$. We may try to compare this integral with $\int_0^{\pi} \sin x\, dx$. How are they related? The region considered in the first integral is obtained by shrinking the region considered in the second integral horizontally. The shrinking factor is 1/2. Thus $\int_0^{\pi/2} \sin 2x\, dx = \frac{1}{2}\int_0^{\pi} \sin x\, dx$. We may generalize it as $\int_a^b f(cx)\, dx = \frac{1}{c}\int_{ac}^{bc} f(x) dx$ where $a, b, c > 0$.



Example) Consider $\int_0^1 \sqrt{x^2+1}\, dx$.

(1) Composite function

Can it be interpreted as $\int_a^b \sqrt{u}\, g(u) du$?

If $g(u)$ is complicated, this interpretation does not work.

We guess anti-derivative of $y = \sqrt{x^2+1}$.

Try $y = (x^2+1)^{3/2}$ => $\frac{dy}{dx} = (3/2)(x^2+1)^{1/2}(2x)$

Trouble: We can not divide by 2x such operation is not interchangeable with derivative.

But $\int 2x\sqrt{x^2+1}\,dx$ can be evaluated directly.

Example) Consider $\int x\sqrt{x^2+1}\,dx$.

(2) Formalize the process

we set $u = x^2+1$. We have in mind the integral $\int \sqrt{u}\,du$

$$\int x\sqrt{x^2+1}\,dx = \int x\sqrt{u}\,dx = \int x\sqrt{u}\,\frac{dx}{du}du = ?$$

$$\frac{du}{dx} = 2x \quad => \frac{dx}{du} = \frac{1}{2x}$$

$$intx\sqrt{u}\,\frac{1}{2x}\,du = \int \frac{1}{2}\sqrt{u}\,du$$

=> $\int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du$

Now we summarize the above observation as follows:

---

2. Substitution Rule (Change of variable formula). Suppose that g is differentiable and monotone, that is, $g' \neq 0$ over an open interval including $[a, b]$, g' is continuous over $[a, b]$, and f is continuous on $[g(a), g(b)]$. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example) Evaluate $\int (2x+1)^3\,dx$.

Set $u = 2x+1$. Then $\frac{du}{dx} = 2$. We have

$$\int u^3 \frac{1}{2}\frac{du}{dx}dx = \frac{1}{2}\int u^3 du = \frac{1}{2}\times\frac{1}{4}u^4 + C = \frac{1}{8}(2x+1)^4 + C$$

Example) $\int \frac{x}{\sqrt{4-x^2}}dx$

$\int (4-x^2)^{-1/2}\, x\, dx$ set u=$4-x^2$ => $\frac{du}{dx}=-2x$

$intu^{-1/2}\frac{1}{2}(2x)dx = \int u^{-1/2}\frac{1}{2}(-1)\frac{du}{dx}dx = -\frac{1}{2}intu^{-1/2}du$

$$=-\frac{1}{2}2u^{1/2}+C=-\sqrt{u}+C=-\sqrt{4-x^2}+C$$

Example) substitution for definite integral

$\int_1^2 (2x+1)^3\, dx$

Set $u=2x+1$. Then $\frac{du}{dx}=2$.

$1 \le x \le 2$ $=> u=2x+1 => 2(1)+1 \le u \le 2(2)+1 => 3 \le u \le 5$

x=1 => u=2(1)+1=3

x=2 => u=2(2)+1=5

We have

$$\int_1^2 u^3\frac{1}{2}\frac{du}{dx}dx = \frac{1}{2}\int_{u=3}^{u=5}u^3du = \frac{1}{8}u^4\Big|_{u=3}^{u=5} = \frac{1}{4}(5^4-3^4)$$

Example) $\int_0^1 \frac{x}{\sqrt{4-x^2}}dx$

set u=$4-x^2$ => $\frac{du}{dx}=-2x$

x=0 => u=4-(0)^2 = 4

x=1 => u=4-(1)^2 = 3

$$\int_0^1 u^{-1/2}\frac{1}{2}(2x)dx = \int_{x=0}^{x=1}u^{-1/2}\frac{1}{2}(-1)\frac{du}{dx}dx$$

$$=-\frac{1}{2}\int_{u=4}^{u=3}u^{-1/2}du = \frac{1}{2}\int_3^4 u^{-1/2}du$$

$$= \sqrt{u}\, RIGHT|_{u=3}^{u=4} = \sqrt{4}-\sqrt{3}$$

Here $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ if a< b

Example) Evaluate $\int_2^3 x^3\sqrt{x^2+1}\,dx$. Take $u=x^2+1$, $\frac{du}{dx}=2x$. Then $2\le x\le 3$ is transformed to $2^2+1=5\le u\le 3^2+1=10$. We have

$$\int_2^3 x^3\sqrt{x^2+1}\,dx=\int_{u=5}^{u=10}\sqrt{u}\,x^2xdx=\int_5^{10}\sqrt{u}\,x^2\frac{1}{2}du$$

$$=\frac{1}{2}\int_5^{10}\sqrt{u}\,(u-1)du=\frac{1}{2}\int_5^{10}u^{3/2}-u^{1/2}du$$

$$=(1/2)(2/5u^{5/2}-2/3u^{3/2})|_5^{10}=(1/5)(10^{5/2}-5^{5/2})-(1/3)(10^{3/2}-5^{3/2})$$

2. Application of substitution rule - First order DE : separable type

(1) What is Differential Equation?

Example) If a ball is shot upward with a velocity 20m/sec, what is the velocity after t seconds?

Solution) v=velocity of the ball after t seconds.

Newton's equation of motion => $\frac{dv}{dt}=-9.8$

where number 9.8 $m/\sec^2$ is gravitational constant.

Velocity function $v=v(t)$ is an anti-derivative of -9.8.

=> $v(t)=-9.8t+C$.

We can understand it as follows: $\int\frac{dv}{dt}dt=\int-9.8dt$. Then we have $v(t)=\int-9.8dt=-9.8t+C$.

How can we determine C?

Initial velocity is $v(0)=20$, thus $v(0)=20=-9.8(0)+C$. Then C=20.

Consequently velocity function $v(t)=-9.8t+20$.

You can also determine when the ball hit the ground if you know the initial height of the ball. Suppose that the ball is shot from the ground. Differential equation describing the height $h=h(t)$ is $\frac{dh}{dt}=v(t)=-9.8t+20$. Then

$$h(t) = \int \frac{dh}{dt} dt = \int 20 - 9.8t \ dt = 20t - 4.9t^2 + C$$

Here again we need to determine C. It is just $h(0) = C$, initial height of the ball. Since the ball was shot from the ground, C=0. We have $h(t) = 20t - 4.9t^2$.

Can you determine when $h(t) = 0$ again?

(2) First order Differential Equation

General form $\frac{dy}{dt} = F(t, y)$. We search for a certain function $y = g(t)$ such that it satisfies the equation, that is $g'(t) = F(t, g(t))$.

Example) Which function satisfies $\frac{dy}{dt} = t + 1$?

$$y = \int \frac{dy}{dt} dt = \int t + 1 \, dt = \frac{1}{2}t^2 + t + C$$

Example) Solve $\frac{dy}{dt} = y$

Note that $y = e^t$ satisfies the relation since its derivative is itself. In general $y = Ce^t$ satisfies the equation. (We will learn this later)

First order DE has a form

$$\frac{dy}{dt} = f(t, y)$$

in general

we search for a function $y = \Phi(t)$ such that

$$\frac{d\Phi}{dt} = f(t, \Phi(t))$$

Note that here Φ', rate of change of Φ depend on Φ itself as well as t

Example) Population change of a certain bacteria in a lab is described by following DE

$$\frac{dy}{dt} = ky^2$$

where y represents the population as a function of time t (in days) and k is a positive constant. Equation means that rate of increase of population is proportional to present population squared.

Since population is squared, the population increases very fast. Our goal is to find a function which satisfies this equation.
Formally we write over equation as

$$\frac{1}{y^2}dy = k\,dt$$

where LHS depends only on y
and RHS depends only on t
we integrate both sides

$$\int \frac{1}{y^2}dy = \int k\,dt$$

Then

$$-\frac{1}{y} = kt - C$$

Thus

$$y = \frac{1}{C - kt}$$

Suppose $k = 0.2$ and initial population is 10.
Then

$$y = \frac{1}{C - 0.2t}$$

and

$$y(0) = \frac{1}{C} = 10$$

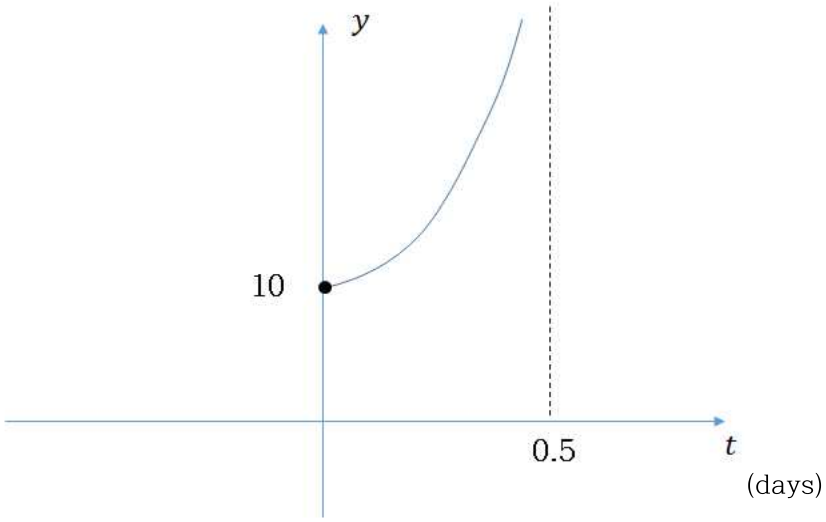
$$C = 0.1$$

Thus

$$y = \frac{1}{0.1 - 0.2t}$$

$$= \frac{5}{0.5 - t}$$

Below is its graph



We can see that the population becomes very large before half day passes.

We say that first order DE is separable if it has a form

$$\frac{dy}{dt} = A(t)B(y)$$

Where RHS is a product of a function of t and a function of y

It can be solved formally by separating the terms according to variables t and y.

$$\frac{1}{B(y)}dy = A(t)dt$$

Where LHS depends only on y and RHS depends only on t.
We integrate both sides

$$\int \frac{1}{B(y)}dy = \int A(t)dt$$

Question: How can the above procedure be justified?

From $\frac{dy}{dt} = A(t)B(y)$, we get

$$\frac{1}{B(y)}\frac{dy}{dt} = A(t)$$

suppose $y = \Phi(t)$ is its solution

Then

$$\frac{1}{B(\Phi(t))}\Phi'(t) = A(t)$$

Now both sides have the same and derivative, that is indefinite integral.

$$\int \frac{1}{B(\Phi(t))}\Phi'(t)\,dt = \int A(t)\,dt$$

<u>By applying substitution rule with $y = \Phi(t)$</u>, the LHS integral (   ) to

$$\int \frac{1}{B(y)}dy$$

Thus

$$\int \frac{1}{B(y)}\,dy = \int A(t)\,dt$$

Example) We take a different model describing population of a bacteria considered in previous example.

$$\frac{dy}{dt} = k\sqrt{y}$$

In this model, the population increase more slowly.

Since it is separable, we have

$$\int \frac{1}{\sqrt{y}}dy = \int k\,dt$$

$$2\sqrt{y} = kt + C$$

$$\sqrt{y} = \frac{k}{2}t + C$$

$$y = (\frac{k}{2}t + C)^2$$

To determine C,

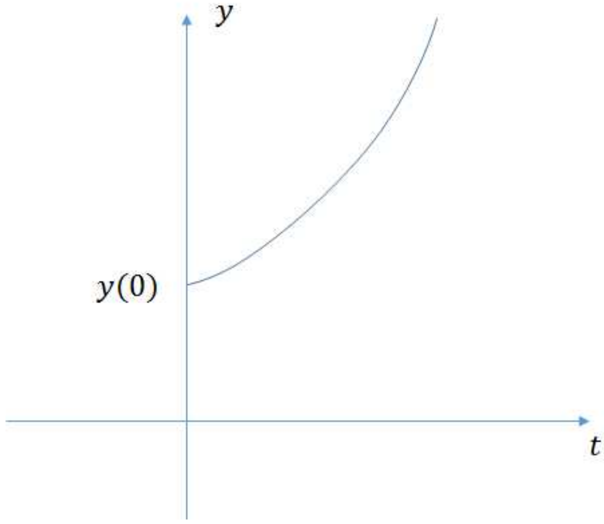
$$y(0) = C^2 \quad C = \pm\sqrt{y(0)}$$

Since $\sqrt{y(0)} = C > 0$

C can take only positive number.

Thus

$$y(t) = (\frac{k}{2}t + \sqrt{y(0)})^2$$



Since y increases slowly there is no blow-up time which appeared in the quadratic model.