Module Optimization problem (Application of derivative)

1. Optimization problem

Optimization problem is about maximizing or minimizing a quantity of interest under certain constraint.

=> One dimensional optimization problem.

=> Higher dimensional optimization problem needs knowledge on differential calculus on multi-variable functions.

Example 1

A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

We will approach the problem step by step

STEP 1 We understand problem.

First decide the quantity to be maximized or minimized.

=> We want to minimize the metal surface area of the can

(can is cylindrical=> surface area of cylinder).

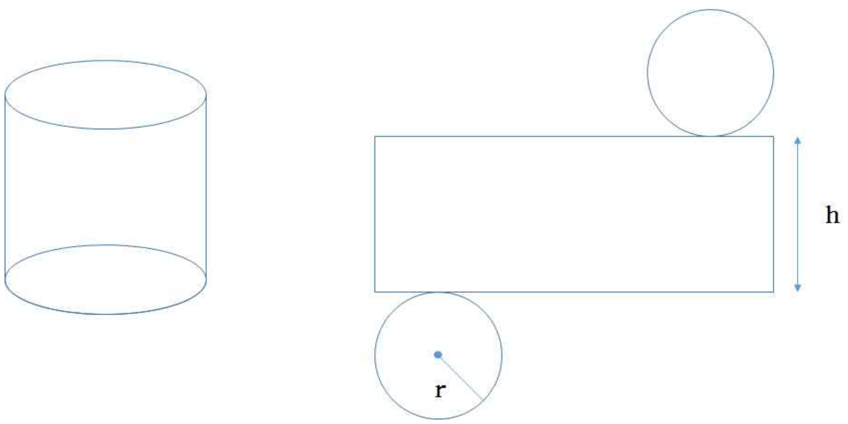
Then we identify the variables which determine the surface area,

=> the size of top/bottom disk

and the height of the can.

*We usually have a constraint => volume of can = 1L

STEP 2 We may need to sketch several diagram.



STEP 3 Introduce the notation for key variables.

height =h

top and bottom disk radius = r

A= surface area of circular cylinder

STEP 4 Remind our goal again: Find h and r which minimize surface area. Write surface area in terms of h and r.

$$A = \text{(area of bottom)} + \text{(area of top)} + \text{(area of rectangle)}$$
$$= 2\pi r^2 + 2\pi rh$$

The constraint in this problem is

$V =$ (area of bottom) • height $= \pi r^2 h = 1000\ cm^3$

We can reduce the number of variable using this constraint as follows

$$A = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$
$$= 2\pi r^2 + \frac{2000}{r} \qquad r > 0.$$

Then A is a function of a single variable. Note that r is positive number and no upper bound. We find the absolute minimum of the function $A(r)$.

Step 5 Find the absolute minimum of the function A(r) for given domain (r>0).

To determine absolute minimum, use I/D test. (=> shpae of the graph A=A(r)

To find critical point, $A' = 4\pi r - \frac{2000}{r^2} = 0$

$$4\pi r^3 = 2000 \qquad r^3 = \frac{500}{\pi}$$

$r = \sqrt[3]{\frac{500}{\pi}}$ (=> we need to check if A has absolute minimum at this number!)

We apply first derivative test to see whether it is local maximum or minimum.

Since $A' = \frac{4\pi r^3 - 2000}{r^2}$, the sign of A' is negative if $r < \sqrt[3]{\frac{500}{\pi}}$ and positive if

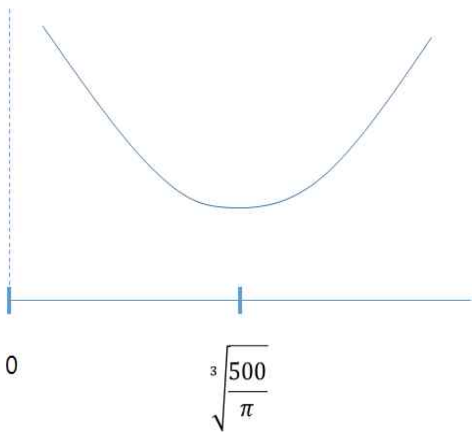
$r > \sqrt[3]{\frac{500}{\pi}}$. Thus A has local min

Note that

$A' > 0 \quad r > \sqrt[3]{\frac{500}{\pi}}$ (numerator of A' is increasing function

$A' < 0 \quad r < \sqrt[3]{\frac{500}{\pi}}$

It implies that A has abs min at $r = \sqrt[3]{\frac{500}{\pi}}$ cm

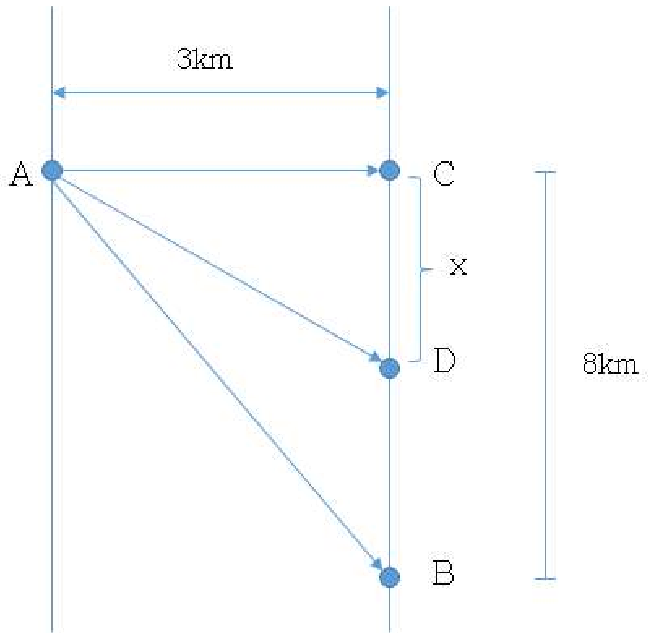


thus, $h = \frac{1000}{\pi(\frac{500}{\pi})^{\frac{2}{3}}} = \frac{(500)^{\frac{1}{3}} \times 2}{\pi^{\frac{1}{3}}}$

$= 2\sqrt[3]{\frac{500}{\pi}}$ cm

Example 2

A man launches his boat from point A on a bank of a straight river, 3km wide. He wants to reach point B, 8km down stream on the opposite bank as quickly as possible. He can row 6km/h on the river and run 8km/h on the bank.



① Goal : minimize time of traveling from A to B

② unknown: Landing point (path)

known: Speed on river and on bank

Introduce the points on the diagram:

C: opposite point of A

D: landing point

③,④ primary equation

traveling time $T = T_{AD} + T_{DB}$

$$time = \frac{distance}{speed}$$

$$T_{AD} = \frac{AD}{6} \quad T_{DB} = \frac{DB}{8}$$

secondary, coordinate for the point D = x (variable)

$$AD = \sqrt{9+x^2}$$

$$DB = 8 - x$$

⑤ $T = \dfrac{\sqrt{9+x^2}}{6} + \dfrac{8-x}{8} \quad 0 \le x \le 8$

(Range for x: why is it that?)

⑥ Determine where T has absolute minimum.

Use closed interval method or I/D test?

critical number $\frac{dT}{dx} = \frac{1}{6}\frac{1}{2}(9+x^2)^{-\frac{1}{2}}(2x) - \frac{1}{8}$

$\frac{x}{6\sqrt{x^2+9}} = \frac{1}{8}$

$4x = 3\sqrt{x^2+9}$

$16x^2 = 9(x^2+9)$

$7x^2 = 81$

$x = \frac{9}{\sqrt{7}}$ (< 9/2=4.5 < 8) it belongs to [0, 8]

We compare T values at two end points and one critical point.

$T(0) = \frac{3}{6} + 1 = 1.5$

$T(\frac{9}{\sqrt{7}}) = \frac{\sqrt{\frac{81}{7}+9}}{6} + 1 - \frac{9}{8\sqrt{7}}$

$= \frac{1}{\sqrt{7}}\frac{\sqrt{81+63}}{6} + 1 - \frac{9/8}{\sqrt{7}} = (2 - \frac{9}{8})\frac{1}{\sqrt{7}} + 1 = \frac{\sqrt{7}}{8} + 1 \approx 1.33$

$T(8) = \frac{\sqrt{64+9}}{6} + 0 = \frac{\sqrt{73}}{6} \approx 1.42$

Trouble: We need calculator to evaluate T. => I/D test

Can we show that $x = \frac{9}{\sqrt{7}}$ is absolute minimum point without using calculator?

We can reason the shape of the graph of T as follows: Since there is no other critical points, the function T is simply decreasing or increasing on $(0, 9/\sqrt{7})$ and $(9/\sqrt{7}, 8)$.

Check the sign of $\frac{dT}{dx} = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$

--------0------------ $x = \frac{9}{\sqrt{7}}$ ---4---------8------->

$T'(0) = -1/8 < 0$ => T is decreasing on $(0, 9/\sqrt{7})$.

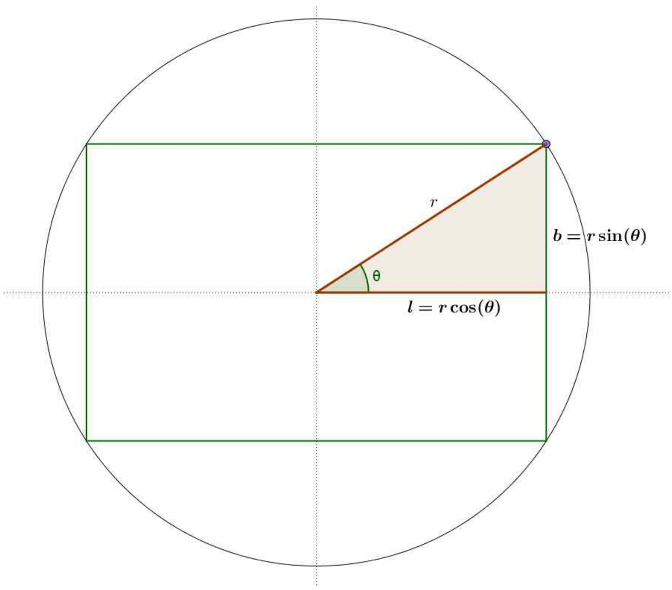
4> $9/\sqrt{7}$ ($16\times 7=112>81$) => $4\in (9/\sqrt{7},8)$

$T'(4)=\frac{4}{6\times 5}-\frac{1}{8}=\frac{2}{15}-\frac{1}{8}=\frac{1}{15\times 8}>0$ => T is increasing on $(9/\sqrt{7},8)$.

=>T has abs min at $x=\frac{9}{\sqrt{7}}$.

Example 3

Find the dimension of the largest rectangle that can be inscribed in a circle of radius r



GOAL: maximize the area of rectangle

variable: width and depth of rectangle

Given condition (constraint): rectangle is inscribed in a circle (radius=r)

To solve this problem, taking a good coordinates is helpful.

Center of circle at the origin on x-y plane. Draw a circle of radius r.

Observation: Rectangle inscribed in the circle is determined by taking a point on the circle on the first quadrant.

The point on the circle is determined by choosing the direction, which is parametrized by angle θ about positive x-axis.

(reduction of number of variables =2 to 1)

=> x-coordinate of the point = $r\cos\theta$

y-coordinate of the point = $r\sin\theta$

=> width of the rectangle = 2 $r\cos\theta$

depth of the rectangle = 2 $r\sin\theta$

=> Area of the rectangle = $4r^2\sin\theta\cos\theta$

$A(\theta) = 4r^2\sin\theta\cos\theta$ ($0 < \theta < \pi/2$) (Why?)

Find absolute maximum of A

(Use double angle formula 2 sin theta cos theta = sin (2 theta))

<= sine addition formula. Please check it)

A= $2r^2\sin 2\theta$ which has abs max at $2\theta = \pi/2 \Rightarrow \theta = \pi/4$

Then largest rectangle should be a square.