Module Natural logarithm and its derivative

- Derivative of inverse function
- Derivative of logarithmic function
- Evaluation of $\lim_{h \to 0} \frac{a^h - 1}{h}$

1. "Derivatives of inverse functions"

$$y = f^{-1}(x) \qquad \frac{dy}{dx} = ?$$

$$f(y) = x \qquad f'(y)\frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(y)}\bigg|_{y=f^{-1}(x)}$$

Exercise) $f(x) = x^2 + 1, \qquad x > 0$

$$y = x^2 + 1 \qquad x = y^2 + 1 \qquad y^2 = x - 1 \qquad y = \pm\sqrt{x-1}$$

$$DOM\ (f) = \{x|x > 0\} \Rightarrow Range(f^{-1}) = \{y|y > 0\}$$

$$y = \sqrt{x-1} = f^{-1}(x)$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{(y^2+1)'} = \frac{1}{2y}\bigg|_{y=\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

Exercise) $f(x) = x^3 + x$

$$f' = 3x^2 + 1 > 0 \qquad \frac{d}{dx}f^{-1}(x)$$

Switch x and y for $y = x^3 + x$ =>

$x = y^3 + y$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(y)}\bigg|_{y=f^{-1}(x)} = \frac{1}{3y^2+1}\bigg|_{y=f^{-1}(x)}$$

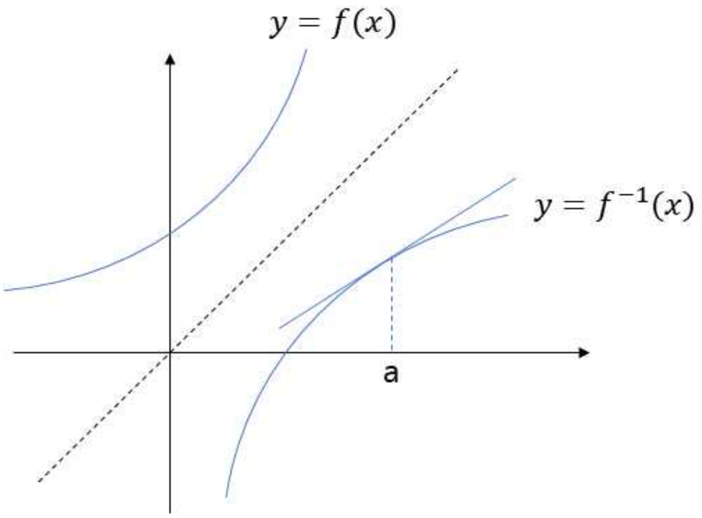
Find the derivative of f^{-1} at 1.

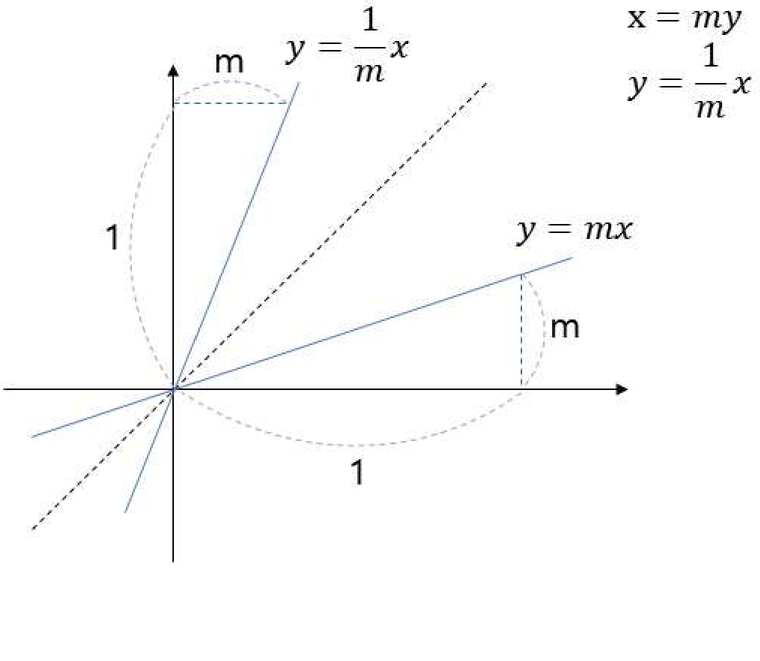
$f(0)=1$

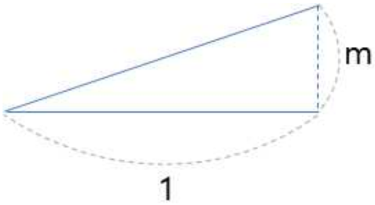
$f^{-1}(1)=0$

$$\left.\frac{d}{dx}f^{-1}(x)\right|_{x=1}=\frac{1}{3f^{-1}(1)^2+1}=\frac{1}{3\cdot(0)+1}=1$$

$$\left.\frac{d}{dx}f^{-1}(x)\right|_{x=a}=\left.\frac{1}{f'(y)}\right|_{y=f^{-1}(a)}$$







$$m = \frac{d}{dx} f^{-1}(x) \Big|_{x=a}$$

$$\frac{1}{m} = \frac{d}{dx} f(x) \Big|_{f^{-1}(a)}$$

$$\Rightarrow \frac{d}{dx} f^{-1}(x) \Big|_{x=1} = \frac{1}{\frac{d}{dx} f(x)} \Bigg|_{x=f^{-1}(a)}$$

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2. "Derivative of logarithmic function"

To study $y = \log_2 x$ consider its inverse function $f(x) = 2^x$

$$y = \log_2 x = f^{-1}(x)$$

(switch x and y, then $x = \log_2 y$ Solve for y $\Rightarrow 2^x = y$)

Definition) The logarithmic function $y = \log_e x$ whose base is natural constant e is called **Natural Logarithm** and denoted by $y = \ln x$.

Finding derivative of natural log

1) $y = \ln x$. Find its inverse. Solve for x. Then $x = e^y$.

$\frac{dx}{dy} = e^y$ and $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$.

(**) $\frac{d}{dx} \ln x = \frac{1}{x}$ (<= Basic fact and used everywhere in calculus!!)

2) Alternative way of finding derivative of natural log

$$\frac{d}{dx}lnx = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{1}{h}ln(1+h/x)$$

$$= \lim_{h\to 0}ln(1+h/x)^{1/h} = \ln\left(\lim_{h\to 0}(1+h/x)^{1/h}\right)$$

(last equality is by continuity of logarithmic function

$$\lim_{s\to a}lnf(s) = \ln(\lim_{s\to a}f(s))$$

set t=h/x. It goes to 0 as h ->0 (x is fixed)

$$\lim_{t\to 0}(1+t)^{1/tx} = \left(\lim_{t\to 0}(1+t)^{1/t}\right)^{1/x} = e^{1/x}$$

(Recall that $e = \lim_{s\to 0}(1+s)^{1/s}$)

=> $\frac{d}{dx}lnx = \ln e^{1/x} = \frac{1}{x}$

3) $\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\log_e x}{\log_e a} = \frac{1}{\ln a}\frac{1}{x}$

(Base change formula $\log_a b = \frac{\log_c b}{\log_c a}$

$$\log_2 3 = \frac{\log_e 3}{\log_e 2} = \frac{\ln 3}{\ln 2}$$

4) Find $\frac{d}{dx}a^x$

$y = a^x$ Solver for x => $x = \log_a y$

=> $\frac{dx}{dy} = \frac{1}{y\ln a}$ => $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{(1/y\ln a)} = y\ln a = a^x\ln a$

4') $\ln a = \lim_{h\to 0}\frac{a^h-1}{h}$

$$\lim_{h\to 0}\frac{2^h-1}{h}=\ln 2=\frac{1}{\log_2 e}<1$$

Exercise) Show that $y=log_2 x$ is concave downward.

Application

What is $\int \frac{1}{x}\,dx$?

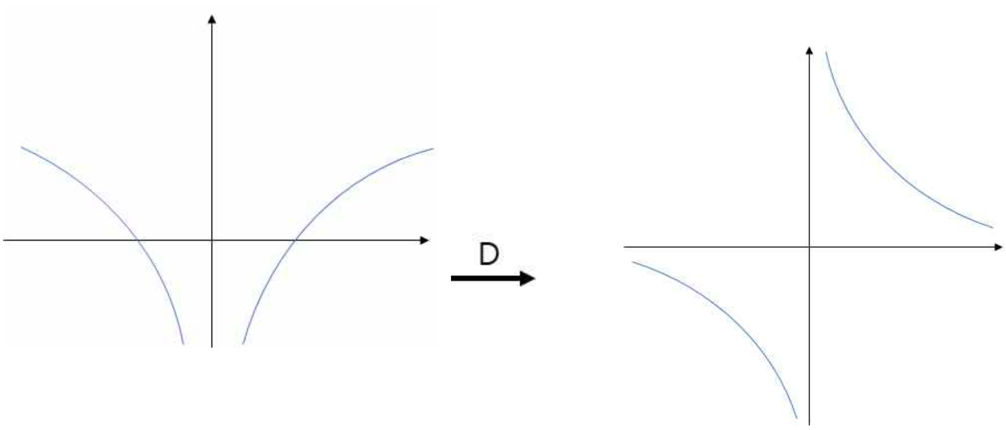
(exceptional case for power rule $\int x^n\,dx=\frac{1}{n+1}x^{n+1}+C\ (n\neq -1)$)

$\frac{d}{dx}lnx=\frac{1}{x}\ \Rightarrow \int \frac{1}{x}dx=\ln x+C$? Correct?

$\frac{d}{dx}\ln\ |x|$ =?

$$\ln\ |x|=\begin{cases}\ln x & x>0\\ \ln(-x) & x<0\end{cases}$$

$$\frac{d}{dx}\ln|x|=\begin{cases}\frac{1}{x} & x>0\\ \frac{1}{-x}(-x)'=\frac{-1}{-x}=\frac{1}{x} & x<0\end{cases}$$



$$\int \frac{1}{x}\,dx=\ln|x|+C$$

Finding $\frac{d}{dx}a^x$ (Two additional methods)

1) Base change for exponential function

$a^x = e^y, \ y = ?$

$\ln a^x = \ln e^y \ \ => x \ln a = y$

$y = a^x = \exp(\ln a^x) = e^{x \ln a}$ (composite function = chain rule for derivative)

$\frac{dy}{dx} = e^{x \ln a} \bullet (x \ln a)'$

$= e^{x \ln a} \bullet (\ln a) = a^x \ln a$

2) Finding $\lim_{h \to 0} \frac{a^h - 1}{h}$ ($\frac{d}{dx} a^x = a^x \left(\lim_{h \to o} \frac{a^h - 1}{h} \right)$

set $u = a^h - 1$ $\qquad h = \log_a (1 + u)$

$$\lim_{u \to 0} \frac{u}{\log_a (1 + u)}$$

$$= \frac{1}{\lim_{u \to 0} \log_a (1 + u)^{\frac{1}{u}}}$$

$$= \frac{1}{\log_a e}$$

$$= \frac{1}{\frac{\ln e}{\ln a}}$$

$$= \ln a$$

$$\frac{d}{dx} a^x = a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$$

$$= a^x \ln a$$

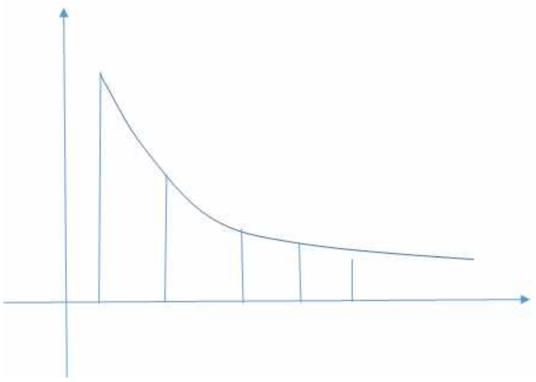
Application

Estimation of $\lim_{h \to 0} \frac{2^h - 1}{h} = \ln 2$

Note that ln 2 = $\int_1^2 \frac{1}{x}dx$

$$\approx R_4 = \frac{1}{4}(f(5/4)+f(6/4)+f(7/4)+f(8/4))$$

$$= \frac{1}{4}(\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{4}{8}) = \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \approx 0.633$$



Check basic question today (Can you answer this question?)

-What is natural logarithm?

-How to find derivative of natural log?

-Using this, how to find derivative of exponential function $y=a^x$?

-What is $\int \frac{1}{x}dx$? How can you justify your answer? (Be careful on domain of its anti-derivative)