Module Method of cylindrical shells

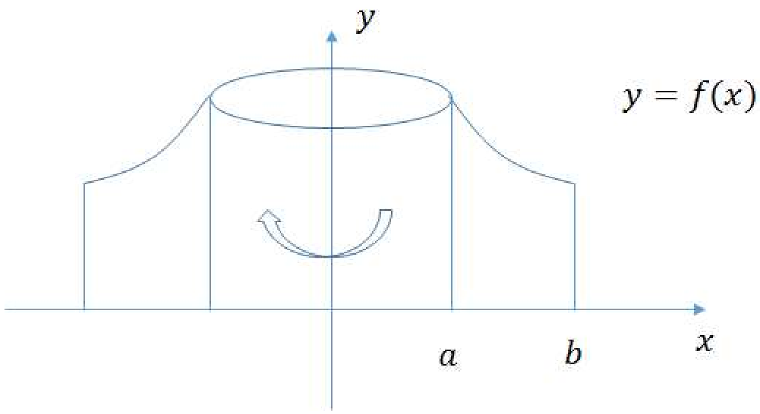
1. Method of cylindrical shells (Stewart section 5.3)

For the solid of revolution about x-axis =>
the area of cross-section perpendicular to x-axis may be natural.

What if we have the solid of revolution about y-axis?
What if the cross-section perpendicular to y-axis is complicated?

=> different method of finding the volume of the solid.

Consider the solid of revolution S about y-axis obtained by rotating the region R below the graph of $y = f(x)$ over $a \le x \le b$.

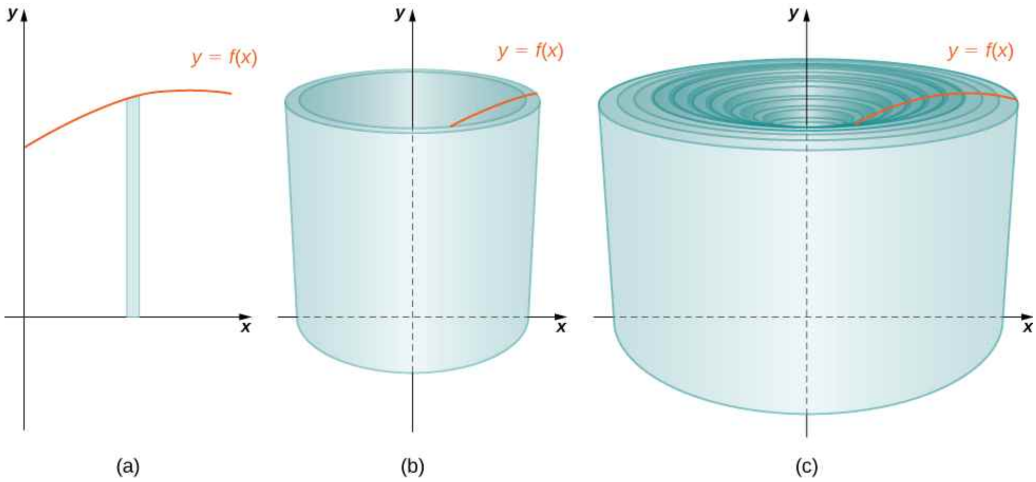


(IDEA)

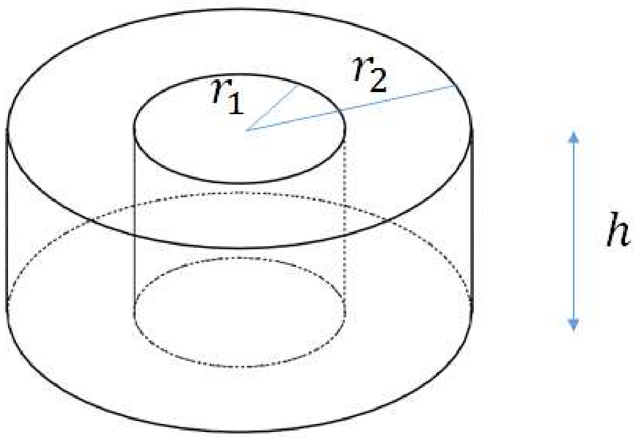
We can divide the region R into several thin vertical strips.

We can view the solid S as an union of solids of revolution of these strips.

Each solid can be approximated by the cylindrical shell if we approximate each strip with rectangular strip and rotate it about y-axis.



The volume of single cylindrical shell is obtained as follows.



Let r_1 be a radius of inner circle and r_2 be a radius of outer circle. Then

$$\begin{aligned} V &= \pi {r_2}^2 h - \pi {r_1}^2 h \\ &= \pi (r_2 + r_1)(r_2 - r_1) h \\ &= 2\pi \left(\frac{r_1 + r_2}{2}\right) \triangle r h \\ &= (2\pi r \triangle r) h \end{aligned}$$

where $r =$ average radius of r_1 and r_2 and $\triangle r =$ thickness of cylindrical shell.

Now we apply this formula to the solid S.

(1) First divide the interval $a \le x \le b$ into n sub-intervals $a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$.

Let $\overline{x_i} = (x_{i-1} + x_i)/2$ be a midpoint of the sub-interval $x_{i-1} \le x \le x_i$ and $\Delta x_i = x_i - x_{i-1}$ (length of ith sub-interval).

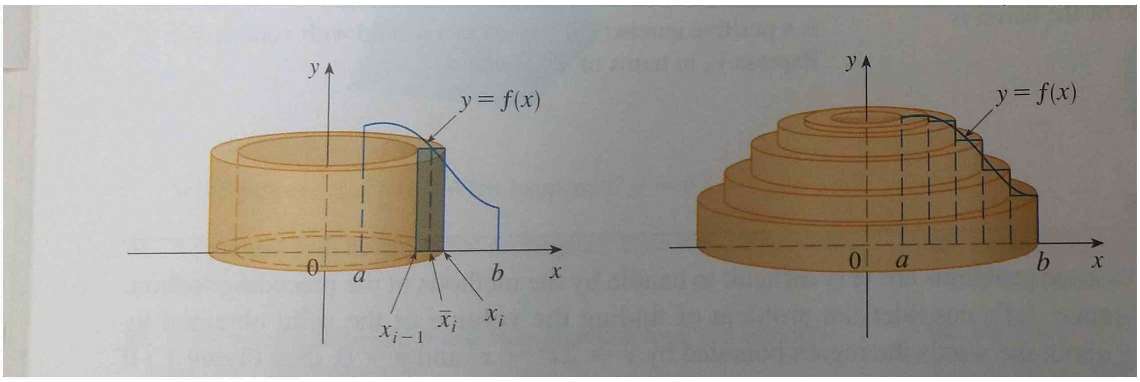
Then <u>the volume of ith cylindrical shell</u> is

$$2\pi \overline{x_i} f(\overline{x_i}) \triangle x_i.$$

(2) The sum of the volumes of cylindrical shells is

$$\sum_{i=1}^{n} 2\pi \overline{x_i} f(\overline{x_i}) \triangle x_i$$

which converges definite integral $\int_a^b 2\pi x f(x) dx$.



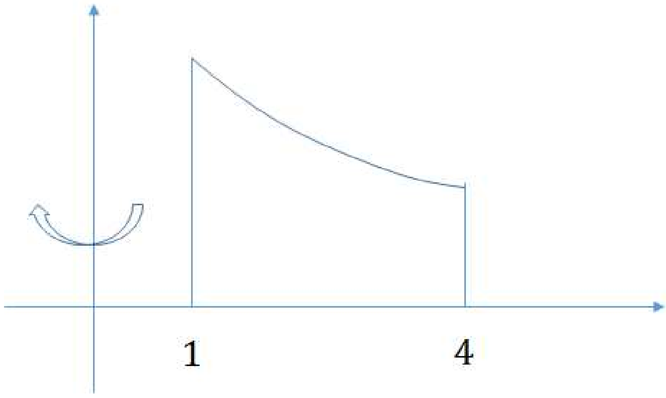
(Formula of volume by cylindrical shells) The volume of the solid S obtained by rotating about y-axis the region under the curve $y = f(x)$ over $a \le x \le b$ is

$$\int_a^b 2\pi x f(x) dx$$

Example) Solid of revolution obtained by rotating $y = \frac{1}{\sqrt{x}}$ about y - axis over

$1 \le x \le 4$

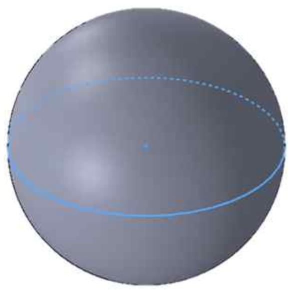
$0 \le y \le \frac{1}{\sqrt{x}}$

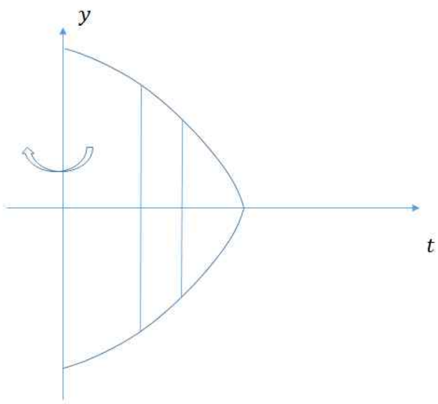


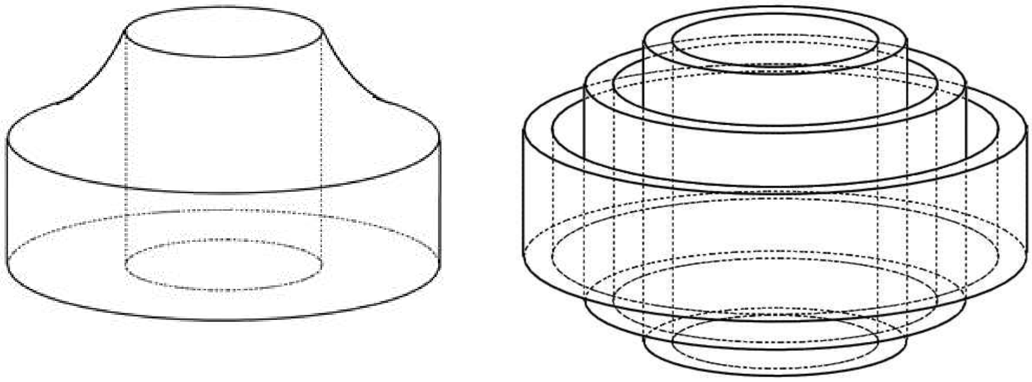
$$V = \int_1^4 2\pi x \frac{1}{\sqrt{x}}\, dx$$

$$= 2\pi \int_1^4 \sqrt{x}\, dx$$

Example) We can find the volume of sphere of radius r using method of cylindrical shell







The sphere of radius r can be considered as the solid of revolution about y-axis, obtained by rotating the region between the curve $y=\sqrt{r^2-x^2}$ and $y=-\sqrt{r^2-x^2}$ over $0 \le x \le r$.

Then the volume by cylindrical shells is

$$V=\int_0^r 2\pi x\ 2\sqrt{r^2-x^2}\,dx$$

$$=4\pi\int_0^r x\sqrt{r^2-x^2}\,dx$$

We compute the integral using substitution rule.

$u=r^2-x^2 \qquad x=0 \Rightarrow u=r^2, \quad x=r \Rightarrow \quad u=0$

$du=-2xdx$

The integral changed into

$$4\pi\int_{r^2}^0 \sqrt{u}\,(-\frac{1}{2})du=2\pi\int_0^{r^2}\sqrt{u}\,du$$

$$=2\pi\frac{2}{3}(r^2)^{\frac{3}{2}}=\frac{4}{3}\pi r^3$$

In general, if the region rotated is between two curves $f(x)$ and $g(x)$ over $a \le x \le b$, then the volume by cylindrical shells

$$V=\int_a^b 2\pi x\,[f(x)-g(x)]\,dx$$

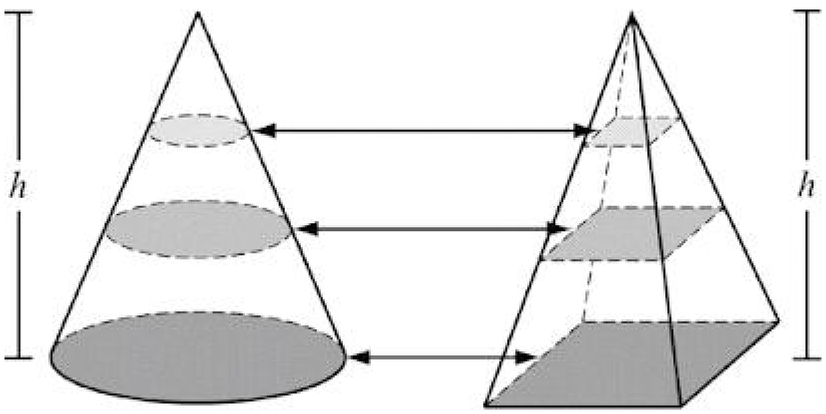
Exercise) Find the volume of a right circular cone using cylindrical shell method.

Exercise) Find the volume of sphere drilled through center with a round hole of radius r (radius of sphere $= R$) using cylindrical shell method.

2. Cavalieri's principle

(Cavalieri(1598-1647) was math professor at Bologna. He was a former student of Galileo Galilei)

If two solids have equivalent bases and if sections parallel to the base and equally distant from them in two solids are equivalent, then the solids have the same volume.



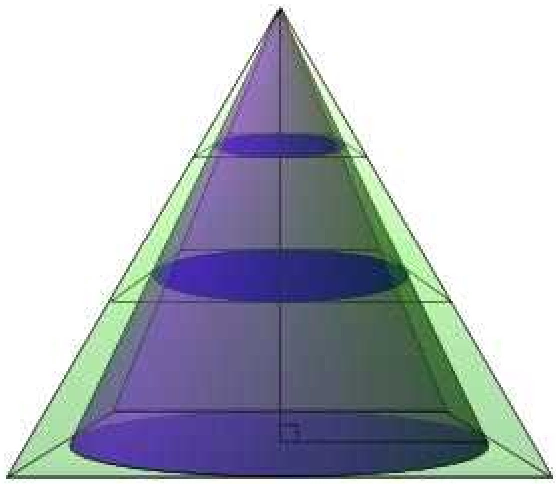
Each corresponding cross sections are equivalent <=> they have the same area

$$V = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx$$

because $A_1(x) = A_2(x)$.

Application

A cone inscribed in a square pyramid



h=height of the pyramid = height of cone

side length of square= d => diameter of cirlce =d => radius is d/2

such a relation holds for every cross section

=> ratio of area of circle to area of square = $\frac{\pi(d/2)^2}{d^2} = \frac{\pi}{4}$

$A_c(x)$= area of cross section of cone at x from the top

$A_p(x)$= area of cross section of pyramid at x from the top

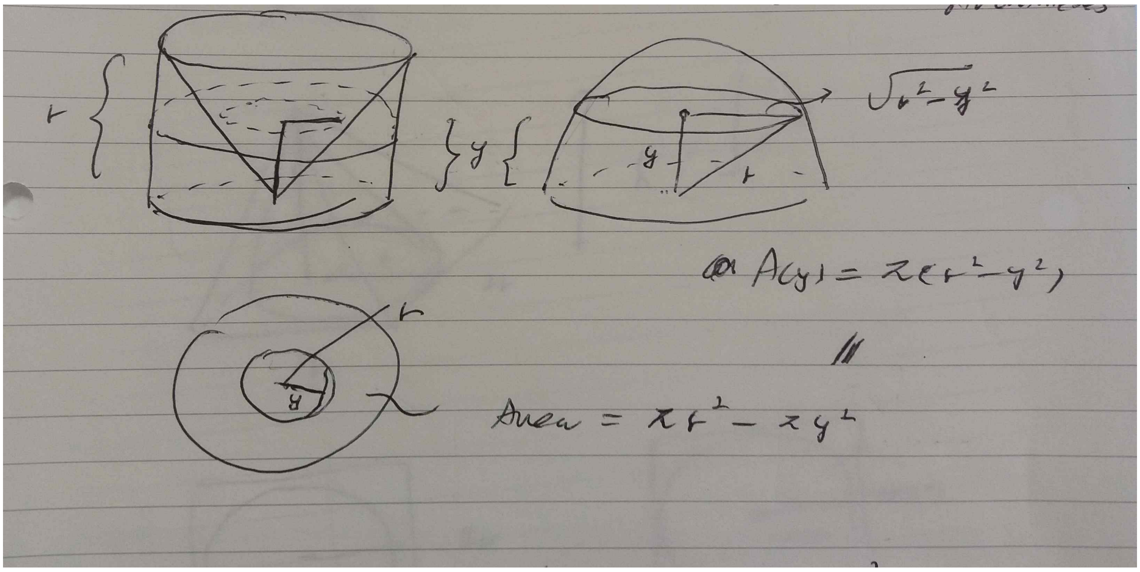
=> $A_c(x)/A_p(x) = \pi/4$

=> Volume of cone = $\int_0^h A_c(x)dx = \frac{\pi}{4}\int_0^h A_p(x)dx$ = $\pi/4$ (Volume of pyramid)

=> $\frac{\pi}{4} \times \frac{1}{3}d^2h = \frac{1}{3}\pi(d/2)^2h$

We can deduce the volume of cone from the volume of pyramid

Archimedes' method of finding volume of sphere (his lost book <The method> was found at Istanbul in 1906)



(IDEA) Construct a simple solid whose cross section is equivalent to the cross section of upper hemisphere.

Cavalieri's principle => volume of hemisphere = volume of (cylinder − cone)

= $(\pi r^2)r - \frac{1}{3}\pi r^2 r = \frac{2}{3}\pi r^3$ => volume of sphere (radius r) = $\frac{4}{3}\pi r^3$