Module Logarithmic functions-basics

- Logarithmic functions
- Inverse function

1. Basics of Logarithms

(1) Definition

Key information for big number such as 1,000,000 => number of digits

Exponent form => 1,000,000=10^6

a: positive number, $a \neq 1$

Logarithm with base a, denoted with $\log_a$, is defined by

$\log_a x = y \Leftrightarrow a^y = x$

So, $\log_a x$ is the exponent to which the base a must be raised to give x

$$\log_a x = y \qquad a^y = x$$

Example)

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2 (\frac{1}{8}) = 3$	$2^{-3} = \frac{1}{8}$
$\log_5 5 = r$	$5^r = 5$

Properties

1. $\log_a 1 = 0 \quad (\because a^0 = 1)$
2. $\log_a a = 1 \quad (\because a^1 = a)$
3. $\log_a a^x = x \quad (\because a^x = a^x)$
4. $a^{\log_a x} = x$

Exercise) Evaluate following numbers

1. $\log_5 1 =$
2. $\log_5 5 =$
3. $\log_5 5^8 =$
4. $5^{\log_5 12} =$

(2) Laws of logarithms

$a > 0,\ a \neq 1,\ A, B(real\ number, A, B > 0)$

1. $\log_a AB = \log_a A + \log_a B$
2. $\log_a \frac{A}{B} = \log_a A - \log_a B$
3. $\log_a A^c = c \log_a A$

Proof (for 1st law)

Let $u = \log_a A,\ v = \log_a B$

In exponent form

$a^u = A, \quad a^v = B$

$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v}) = u + v = \log_a A + \log_a B$

Exercise)

1. $\log_4 2 + \log_4 32 = ?$ $(ans : 3)$ Use 1st law
2. $\log_2 80 - \log_2 5 = ?$ $(ans : 4)$
3. **Expand** $\log_3(\frac{ab}{\sqrt[3]{c}})$
4. **Expand** $\log_5(x^3 y^6)$
5. **Combine** $3\log_2 x + \frac{1}{2} log_2(x+1)$ **into a single log .**
6. Explain why followings are not true

$a)\ \frac{\log_3 6}{\log_3 2} = \log_3(6/2) = \log_3(3) = 1$

$b)(\log_2 x)^3 = 3\log_2 x$

Def) Common Logarithm

The logarithm with base 10 is called the common logarithm and is denoted by

$$\log x = \log_{10} x$$

(scientific calculator are equipped with **LOG** key **that** directly gives **values** of common logarithms)

Exercise) Using calculator evaluate

$\log 2,\ \log 3,\ \log 5,\ \log 7$

Remark: common log is not used in calculus! inconvenient for finding derivative of logarithmic function (also exponential function) => Use so called natural log!!

Example)

$2^x = 7 \quad Find\ x$ (using common log)

Take common log of both sides

$$\log 2^x = \log 7 \qquad (Law\ 3)$$

$$x \log 2 = \log 7$$

$$x = \frac{\log 7}{\log 2} \qquad (solve\ for\ x)$$

$$Evaluating\ it\ using\ calculator$$

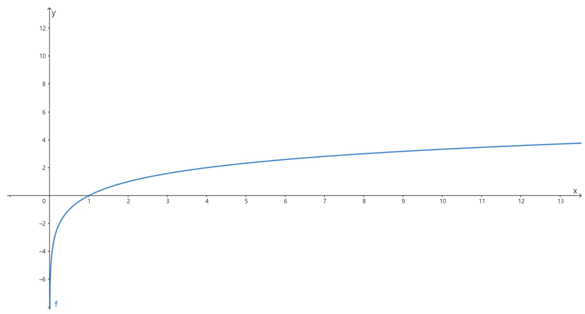
$$x \approx \frac{0.845}{0.301} \approx 2.8$$

2. Logarithmic function

$y = \log_a x$

Example) $y = \log_2 x$ Domain =$(0, \infty)$, Range = R

x	$\log_2 x$	x	$\log_2 x$
1=2^0	0	$\frac{1}{2}$ =2^-1	-1
2=2^1	1	$\frac{1}{4}$ =2^-2	-2
4=2^2	2	$\frac{1}{8}$ =2^-3	-3
8	3		



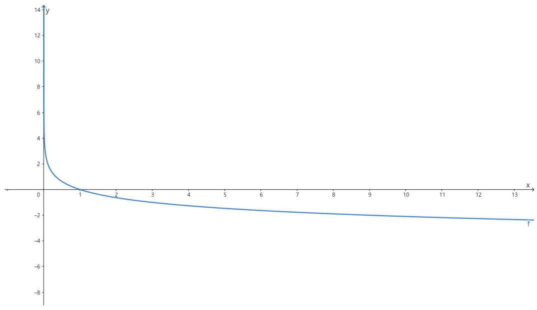
$$\lim_{x \to 0^+} \log_2 x = \lim_{N \to \infty} \log_2 \frac{1}{2^N} = \lim_{N \to \infty} -N = -\infty$$

$x = 0$ is vertical asymptote

- Increasing
- x-intercept : (1,0)
- $\lim_{x \to \infty} \log_2 = \infty$
- Continuous
- one to one

$y = \log_{\frac{1}{3}} x$

x	$\log_{1/3}x$	x	$\log_{1/3}x$
1	0	$\frac{1}{3}$	1
3	-1	$\frac{1}{9}$	2
9	-2	$\frac{1}{27}$	3
27	-3		



$$\lim_{x \to 0^+} \log_{\frac{1}{3}} x = \lim_{N \to \infty} \log_{\frac{1}{3}} \frac{1}{3^N} = \lim_{N \to \infty} N = \infty$$

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3. Inverse function

$f : X \to \mathrm{Y}$

$x \to f(x)$

Definition)

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (f is one to one)

Definition) f is onto $\Leftrightarrow$ $f(X) = Y$

where f(A) is image of f, that is the set of all function values

Example) $f(x) = x^2 + 1$

$f : R \to \{y | y \geq 1\}$ It is not one to one but onto (f(1)=f(-1))

Example) $g(x) = \sqrt{x}$

g: set of positive real numbers ->set of positive real numbers

=>one to one

If $f: X \rightarrow Y$ is one to one and onto, then for each $b \in Y$, Then **exists unique** $a \in X$ **s,t** $f(a) = b$

Definition) Inverse function of f : X -> Y (one to one and onto)

$f^{-1}:\ Y \rightarrow X$

$y \rightarrow f^{-1}(y)$

$f(f^{-1}(y)) = y$

Domain of $f^{-1} =$ Range of f

Range of $f^{-1} =$ Domain of f

Example) $f(x) = 2x + 3 = y$ Here f: R -> R is one to one and onto

$x = \frac{y-3}{2}$

$y \rightarrow \frac{y-3}{2}$ inverse function of f

$x \rightarrow 2x+3$ $\quad$ $y \rightarrow \frac{y-2}{2}$

$1 \rightarrow 2+3=5$ $\quad$ $5 \rightarrow \frac{5-3}{2} = 1$

Example)

$f(x) = x^2$ $\{x | x \geq 0\}$(Domain)

f: set of all nonnegative real numbers -> set of all nonnegative real numbers

=> f is one to one and onto.

To find inverse function solve $y = x^2$ for x

=> $x = \sqrt{y}$ $\qquad$ $\{y | y \geq 0\}$ defines inverse function

f^{-1} (y) = sqrt{y}

Example)

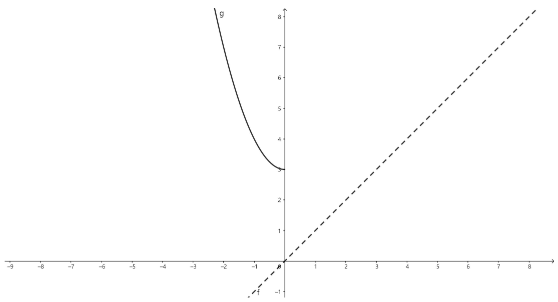
$y = x^2 + 3$ $\qquad$ $\{x | x < 0\} \rightarrow \{y | y > 3\}$

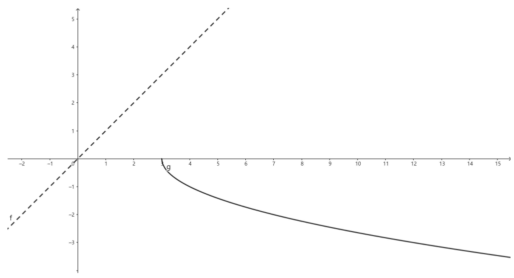
switch x and y => $x = y^2 + 3$ (use of x as domain variable is conventional)

Solve for y, $y = \pm\sqrt{x-3}$

$f^{-1}: \{x | x > 3\} \rightarrow \{y | y < 0\}$

$y = -\sqrt{x-3} = f^{-1}(x)$





Here dotted line is y=x.

Switching x and y => reflection of graph y=f(x) about the line y=x

=> x=f(y) <=> $y = f^{-1}(x)$

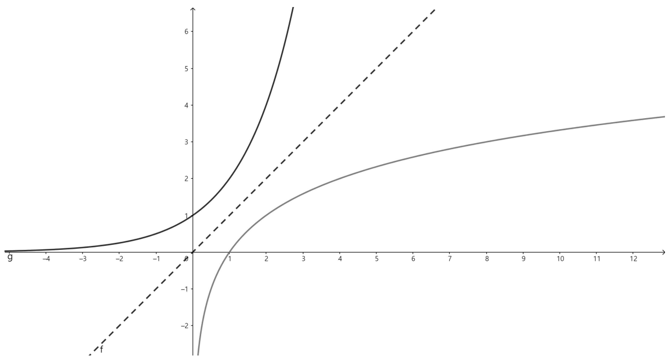
Example

$y = 2^x \quad R \to \{y > 0\}$. Find its inverse function

=> switch x and y

$x = 2^y$ Then solve for y. $\quad y = \log_2 x$ (Inverse function of $y = 2^x$)

$\{x > 0\} \to R.$



-Graph of exponential function and its inverse function (=logarithmic function)

In general

$y = a^x$ => its inverse function : switch x and y $x = a^y$ and

solve for y => $y = \log_a x$ (by definition of log)

$f^{-1} \circ f(x) = \log_a a^x = x \quad (f(x) = a^x)$

$f \circ f^{-1}(y) = a^{\log_a y} = y$