Module Limit of functions

1. Why do we need real numbers?

Natural number => Integers => Rational number =>?

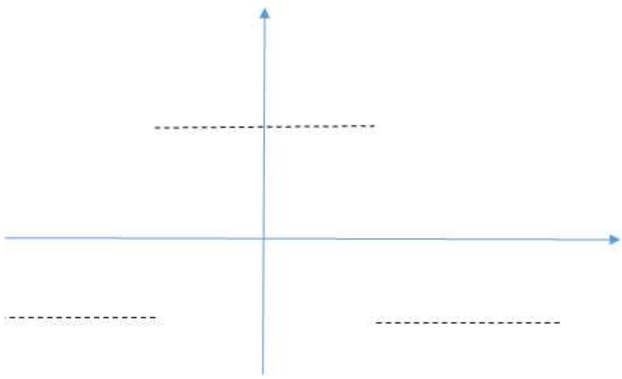
Q = set of all rational numbers

Example) $f : Q \to Q$ $\quad f(x) = \begin{cases} 1, & \text{if } x^2 < 2 \\ -1, & \text{otherwise} \end{cases}$

Is it continuous everywhere?

It has a jump at x for which $x^2 = 2$. Such x is irrational, which is not an element of the domain of function. Thus there is no point on the domain, at which f has a jump.

What is wrong with this?



Completeness of Real number system.

Def) (x_n): sequence of numbers. $x_1, x_2, x_3, \ldots$

Example) Fibonacci sequence. $x_1 = 1,\ x_2 = 1,\ x_n = x_{n-1} + x_{n-2}\ \ (n \geq 3)$

$$x_3 = 2, x_4 = 3, x_5 = 5, \ldots$$

Def) Limit of sequence

$$\lim_{n \to \infty} x_n = x$$

if $|x_n - x|$ becomes smaller and smaller as n becomes larger and larger.

Here number x is called the limit of the sequence (x_n)

Example) $x_n = \frac{n}{n+1}$ Then $\lim_{n \to \infty} \frac{n}{n+1} = 1$ because $\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1}$ becomes smaller and smaller as n becomes larger and larger.

Def) A given number system is said to be <u>complete</u> if every convergent sequence (precisely speaking Cauchy sequence) in this number system has a limit inside this number system.

Example) $\boldsymbol{Q}$ is not complete. A sequence $x_1 = 1,\ x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}),\ n \geq 1$ is convergent, that is Cauchy, which means that $|x_n - x_m| \to 0$ as n, m $\to \infty$. But its limit is $\sqrt{2}$, which is not element of $\boldsymbol{Q}$.

Set of real numbers is constructed by adding all the limits of rational convergent sequences. It is known that set of all real numbers is complete.
=> We have no trouble when we handle convergent sequence. Limiting process works very well.

2. Limit of functions

Examples of elementary functions

(1) Polynomials

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

(2) Rational functions

$\frac{P(x)}{Q(x)}$ where $P(x)$, $Q(x)$ are polynomials

(3) Algebraic functions

$f(x) = \sqrt{x^2+3}$, $g(x) = \frac{\sqrt[3]{x}}{1+x^2}$

(4) Transcendental function

$f(x) = 3 + 2\sin 5x$, $g(x) = 2^x$, $h(x) = \frac{1}{\log_3 x}$

◉ **Behavior of function values near a certain point**

(ref: Stewart (7th edition), section 1.5/1.7)

Example $f(x) = \frac{1}{x}$, Domain = set of all real numbers except 0

Question: How the values of $f(x) = \frac{1}{x}$ change as x approaches 0?

x	f(x)=1/x	x	f(x)=1/x
0.1	10	-0.1	-10
0.01	100	-0.01	-100
0.001	1000	-0.001	-1000

Behavior of $f(x) = \frac{1}{x}$ depends on the direction x approaches 0.

Definition) $\lim_{x \to a} f(x) = L$

if $|f(x) - L|$ becomes smaller and smaller as $|x-a|$ becomes arbitrarily small.

Example) $\lim_{x \to 2} 3x - 1 = 5$ because $|3x-1-5| = |3x-6| = 3|x-2| \to 0$ as $|x-2| \to 0$.

Example) $\lim_{x \to 1} \frac{x^2-1}{x-1}$ =?

Note that given function is not defined at x=1

x	$\frac{x^2-1}{x-1}$	x	$\frac{x^2-1}{x-1}$
1.1	2.1	0.9	1.9
1.01	2.01	0.99	1.99
1.001	2.001	0.999	1.999

We can see that $\frac{x^2-1}{x-1}$ seems to approach 2 as |x-1| becomes smaller and smaller.

$\left| \frac{x^2-1}{x-1} - 2 \right| = \left| \frac{x^2-1-2(x-1)}{x-1} \right| = \left| \frac{(x-1)^2}{x-1} \right| = |x-1| \to 0$ as |x-1| ->0

Definition) One sided limit

Left-hand limit

$\lim_{x \to a-} f(x) = L$ means that $|f(x) - L| \to 0$ as $x < a$ and |x-a|->0

Right-hand limit

$\lim_{x \to a+} f(x) = L$ means that $|f(x) - L| \to 0$ as $x > a$ and |x-a|->0

Example) $f(x) = \begin{cases} x, & x < 1 \\ x^2 + 1, & x > 1 \end{cases}$

$\lim_{x \to 1-} f(x)$= $\lim_{x \to 1-} x = 1$, $\lim_{x \to 1+} f(x)$= $\lim_{x \to 1+} x^2 + 1 = 2$

In this case, we say $\lim_{x \to 1} f(x)$ does not exist.

◉ **Limit laws (ref: section 1.6)**

Suppose that $\lim_{x \to a} f(x) = K$, $\lim_{x \to a} g(x) = L$.

(1) $\lim_{x \to a} f(x) \pm g(x) = K \pm L$

(2) $\lim_{x \to a} \alpha f(x) = \alpha K$ (α: any real number)

(3) $\lim_{x \to a} f(x) g(x) = KL$

(4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{K}{L}$ $(L \neq 0)$

(5) $\lim_{x \to a} \sqrt{f(x)} = \sqrt{K}$

Direct substitution property

(1) If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial, then $\lim_{x \to c} P(x) = P(c)$.

It is consequence of Limit Laws. (Verify why it is true)

(2) For polynomials $P(x)$, $Q(x)$, if $Q(c) \neq 0$, then $\lim_{x \to c} \frac{P(x)}{Q(x)}$ = $\frac{P(c)}{Q(c)}$

Example) $\lim_{x \to 2} 3x^2 - x + 4 = 3(2)^2 - 2 + 4 = 14$

Exercise) Evaluate the limit $\lim_{x \to 1} \frac{x^2+1}{x+3}$

3. Squeeze Theorem

Motivating question :

Suppose that $f(x)$ is defined on $(-1,0) \cup (0,1) = \{x : -1 < x < 1, x \neq 0\}$ and bounded, that is, there exists some non-zero constant M such that $|f(x)| \leq M$ on its domain. What is the limit $\lim_{x \to 0} x^2 f(x)$?

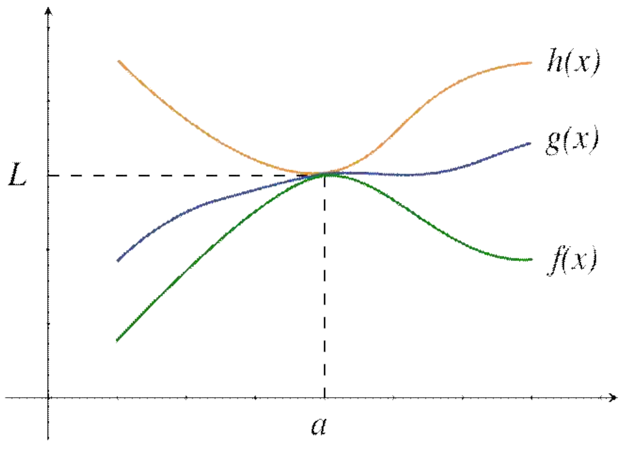
Trouble) We can not use limit law-product case (why?)

Following theorem is useful.

The Squeeze Theorem (or sandwich Theorem)

If $f(x) \leq g(x) \leq h(x)$ over $(a-\epsilon, a) \cup (a, a+\epsilon)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

$\lim_{x \to a} g(x) = L$



Apply squeeze theorem to our case. Since $-M \leq f(x) \leq M$, $-Mx^2 \leq x^2 f(x) \leq Mx^2$.

Sine $\lim_{x \to 0} Mx^2 = \lim_{x \to 0} -Mx^2 = 0$, we can apply squeeze theorem. Thus we have

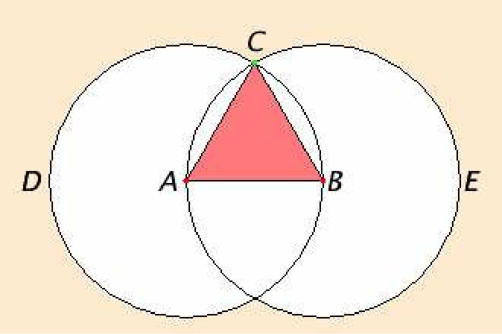
$$\lim_{x \to 0} x^2 f(x) = 0$$

* Squeeze theorem is essential to derive differentiation formula for sine function and cosine function. (We will see it soon)

Continuous function (ref: section 1.8)

Famous first proposition of Euclid's book <<Elements>>

We can construct an equilateral triangle with side of given length using only ruler and compass.



(Suspicious thing) It is not clear whether two circles have an intersection.

Is the following true? (Wired intermediate value theorem?)

$f: \mathbf{Q} \to \mathbf{Q},\ f(x) = x^2 - 2.$

We found that $f(0) = -2 < 0,\ f(2) = 2 > 0$. There exists a number c between 0 and 2 such that $f(c) = 0$.

(That is, doe the graph $y = f(x)$ intersect x-axis somewhere?)

=> same situation with Euclid two circle intersection argument.

=> We need notion describing the case our curve has no hole inside.

Def) A function f is continuous at x=a if $\lim_{x \to a} f(x) = f(a)$

Remark) To discuss continuity of f at x=a, we need to assume that f is defined on $(a-c, a+c)$ for some positive c. Note that $(a-c, a+c)$ has no hole.

Example) If f is polynomial, it is continuous everywhere.

Exercise) $g(x)=\begin{cases}1, & x\ge 1\\ 0, & x<1\end{cases}$. Is g continuous at x=1?

Exercise) $h(x)=\begin{cases}1, & x \text{ is rational}\\ 0, & x \text{ is irrational}\end{cases}$. Where is h continuous?

Remark) Following criterion is useful

$\lim_{x\to a} f(x)=L$ if and only if $\lim_{n\to\infty} f(x_n)=L$ for every sequence (x_n) with $\lim_{n\to\infty} x_n=a$

Example) $\lim_{x\to 2} x^2=4$ <=> $x_n=2+\frac{1}{n}$ which converges 2, $\lim_{n\to\infty}(2+\frac{1}{n})^2=\lim_{n\to\infty}(4+\frac{2}{n}+\frac{1}{n^2})=4+0+0=4$

Application to limit of function

Limit law

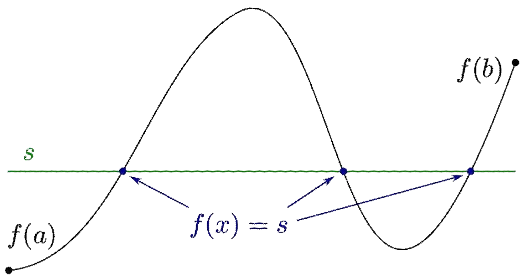
If $\lim_{x\to a} g(x)=L$ and f(x) is continuous at x=L, then $\lim_{x->a} f(g(x))=f(L)$

Example) $\lim_{x\to 1}\sqrt{x^2+3}=\sqrt{\lim_{x\to 1}(x^2+3)}=\sqrt{4}=2$

(You can switch square root with limit)

4. Intermediate Value Theorem

Suppose that f is continuous on the closed interval [a, b] and let N be any number between f(a) and f(b) (We assume that $f(a)\neq f(b)$). Then there exists a number c between a and b such that $f(c)=N$.



(Question) Why is this theorem true?

Topologist describes a reason why this theorem holds as follows: a continuous function sends a connected set into another connected set. (Here [a, b] is connected set).

Application) Show that the equation $x^2 = 2$ has a root between 1 and 2 using intermediate value theorem.

*Next time

Velocity problem and tangent problems => notion of derivative

*If you don't have textbook, you can look up following sections of Strang's book for Today's topics

https://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/

-section 2.6 (Limit of function)

-section 2.7 (Continuity)