Module Integration by parts-application

Recall integration by parts

$\int uv' \, dx = u(x)v(x) - \int u' v \, dx$

Integration by parts-definite Integral version

From derivative of product of two functions

$$\int_a^b \frac{d}{dx}(uv)\,dx = uv\,\Big|_{x=a}^{x=b} = \int_a^b u'v + \int_a^b uv'$$

which gives

$$\int_a^b uv'\,dx = uv\Big|_{x=a}^{x=b} - \int_a^b u'v\,dx$$

Ex)(Case of applying integration by parts twice)

Evaluate $\int_a^b e^x \sin x\,dx$

$u = \sin x \quad v' = e^x$

$u' = \cos x \quad v = e^x$

Let $I = \int_a^b e^x \sin x\,dx$, then

$$I = e^x \sin x\,\Big|_a^b - \int_a^b e^x \cos x\,dx$$

Apply integration by parts again

$$\int_a^b e^x \cos x\,dx = e^x \cos x\,\Big|_a^b - \int_a^b e^x(-\sin x)dx$$

We obtain

$$I = e^b \sin b - e^a \sin a - \left[e^b \cos b - e^a \cos a + I\right]$$

Solve for I, then

$$2I = e^b(\sin b - \cos b) - e^a(\sin a - \cos a)$$

$$I = \frac{1}{2}\left[e^b(\sin b - \cos b) - e^a(\sin a - \cos a)\right].$$

Exercise) Evaluate $\int_{e}^{e^2} (\ln x)^2 \, dx$ (Hint: Apply u=(ln x)^2 and v'=1)

Reduction formula

When we integrate a power of a function, it will be easy to compute the integrals by using reduction formula.

Example) Obtain reduction formula for $\int \sin^n x \, dx$. (sin^n x means (sin x)^n)

$u = \sin^{n-1} x$ $\qquad v' = \sin x$

$u' = (n-1)\sin^{n-2} x \cos x \quad v = -\cos x$

(Apply chain rule to $\frac{d}{dx}(\sin x)^{n-1} =$ derivative of $()^{n-1} \times (\frac{d}{dx} sinx)$

$$\int u v' \, dx = u(x)v(x) - \int u' v \, dx$$

Let $I_n = \int \sin^n x \, dx$. Then

$$I_n = \sin^{n-1} x (-\cos x) - \int (n-1)\sin^{n-2} x \cos x (-\cos x) dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} dx$$

$$= -\sin^{n-1} x \cos x + (n-1)(I_{n-2} - I_n)$$

Solve for I_n, then

$$nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

It is desired reduction formula. If we apply it to compute I_2 *nd I_3, then we have

$$I_2 = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C$$

$$I_3 = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}I_1$$
$$= -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}(-\cos x) + C$$

Example. Show that $J_{2n} = \int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\frac{\pi}{2}$

(Use reduction formula for I_n)

Example. Wallis product formula: $\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots$

Estimation of pi had been important math problem (Finding area of circle and volume of ball)

Ancient Egyptian and Babylonian used very rough estimation.

First breakthrough: Archimedes (method of exhaustion)

John Wallis (1616-1703): notation ∞ was introduced by him. One of early pioneers of study of integration

Claim 1. $\frac{J_{2n+2}}{J_{2n}} = \frac{2n+1}{2n+2}$ (Exercise)

Claim 2. $J_{2n+2} \le J_{2n+1} \le J_{2n}$

$$\int_0^{\pi/2} \sin^{2n+2} x\, dx \le \int_0^{\pi/2} \sin^{2n+1} x\, dx \le \int_0^{\pi/2} \sin^{2n} x\, dx$$

Claim 1 and 2 implies that

$$\frac{J_{2n+2}}{J_{2n}} \le \frac{J_{2n+1}}{J_{2n}} \le 1$$

$$\frac{2n+1}{2n+2} \le \frac{J_{2n+1}}{J_{2n}} \le 1$$

Thus we have $\lim_{n\to\infty} \frac{J_{2n+1}}{J_{2n}} = 1$.

Claim 3

$$J_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{(Exercise)}$$

Now we have

$$\frac{J_{2n+1}}{J_{2n}} = \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdots (2n+1)} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \frac{2}{\pi} \to 1$$

$$\Rightarrow \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\frac{\cdots 2n 2n}{\cdots (2n-1)(2n+1)} \to \frac{\pi}{2}$$

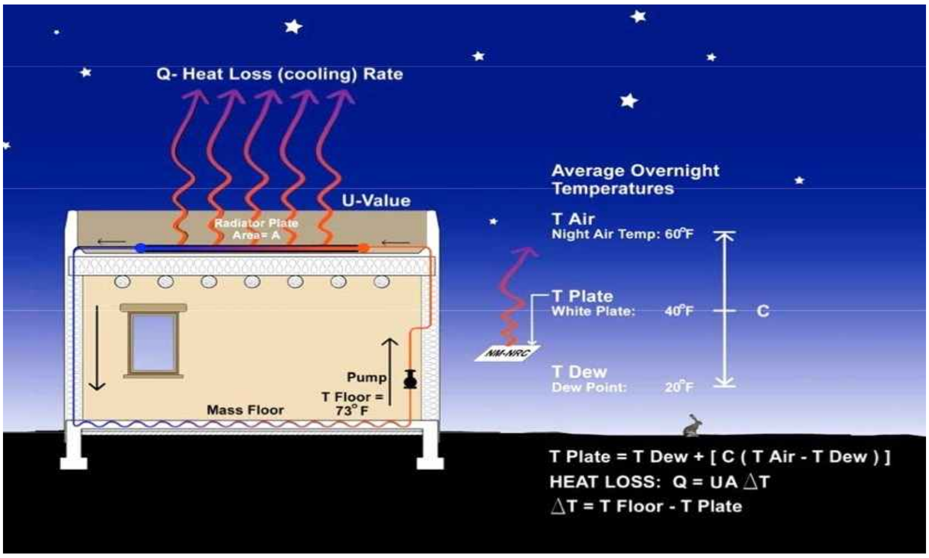
Application of integral $\int e^{kt} \cos\omega t\, dt$, $\int e^{kt} \sin\omega t\, dt$

Problem) In spring time, daily temperature change of outside is big. We want to know how temperature of inside the building changes in response to outside temperature change.

$u(t)$= internal temperature of building. Internal temperature of the building satisfies Newton's cooling law.

$$\frac{du}{dt} = -k[u - T(t)]$$

where $T(t)$ is the surrounding (outside) temperature suppose that $T(t) = T_0 + T_1 \cos\omega t$.



We want to study $u(t)$. The differential equation is not separable any more. We rewrite the equation as follows:

$$\frac{du}{dt} + ku = k\,T(t)$$

To find its solution, first multiply e^{kt} to both sides, then

$$e^{kt}\,(u' + ku) = ke^{kt}\,T(t)$$

Note that the left-hand side (LHS) $= \frac{d}{dt}\,(e^{kt}u) = ke^{kt}u + e^{kt}u'$. Thus our equation changes into

$$\frac{d}{dt}\,(e^{kt}u) = ke^{kt}\,T(t)$$

Integrate both sides then

$$e^{kt}u = \int ke^{kt}\,T(t)dt.$$

We have only to compute the integral of the right-hand side

$$RHS = k\int e^{kt}\,(\,T_0 + T_1\cos\omega t)\,dt$$

$$= kT_0\int e^{kt}\,dt + kT_1\int e^{kt}\cos\omega t\,dt$$

$$= T_0 e^{kt} + kT_1\frac{e^{kt}}{k^2+\omega^2}\,[k\cos\omega t + \omega\sin\omega t] + C$$

(Here evaluation of the second integral is exercise. Use integration by parts twice)
Solve for u, then

$$u(t) = T_0 + T_1 \frac{k}{k^2+\omega^2}[k\cos\omega t + \omega\sin\omega t] + Ce^{-kt}$$

The first part is called **steady state** and the second part is called **transient state.** Note that the transient part $\to 0$ as $t \to \infty$.

To decide constant C, use initial temperature $u(0) = u_0$.

$$u(0) = T_0 + \frac{k^2 T_1}{k^2+\omega^2} + C = u_0$$

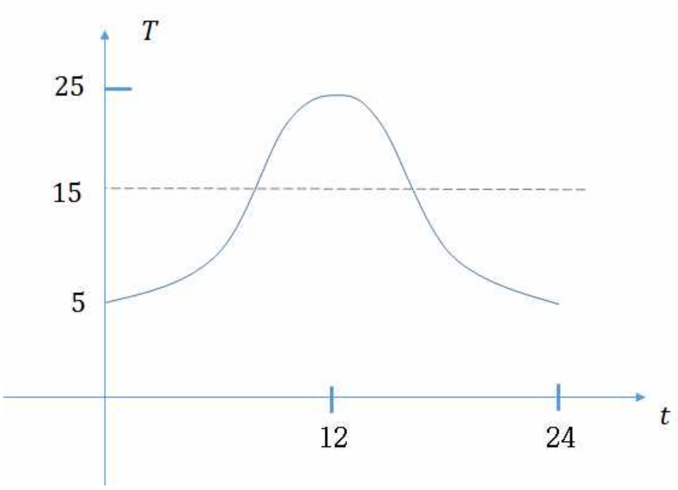
$$C = u_0 - T_0 - \frac{k^2}{k^2+\omega^2} T_1$$

Example) The period of temperature outside of building is 24 hours. Average temperature outside is 15 degree and the highest is 25 degree and the lowest is 5 degree. Assume that k=0.2 where k depends on the insulation or conductivity of the building.

=> $T(t) = T_0 + T_1 \cos\omega t$

$\omega = \frac{\pi}{12}$, $T_0 = 15^\circ C$, $T_1 = -10^\circ C$, $k = 0.2$ (Period of our function is $2\pi/\omega$)

Then the change of the temperature outside is $T(t) = 15 - 10\cos\frac{\pi}{12}t$. Its graph is as follows:

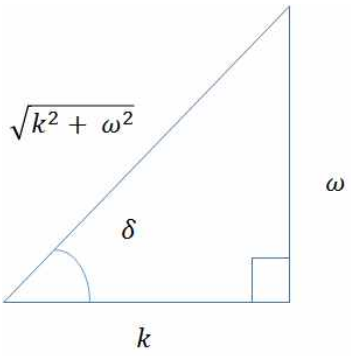


Let us first study steady state solution. We synthesize the sum of sine and cosine functions to a single cosine function. We can find out amplitude of oscillation

$$S(t) = T_0 + \frac{k}{k^2 + \omega^2} T_1 [k\cos\omega t + \omega\sin\omega t]$$

$$= T_0 + \frac{k}{\sqrt{k^2 + \omega^2}} T_1 \cos(\omega t - \delta)$$

(cosine addition formula is used)



where $\cos\delta = \frac{k}{\sqrt{k^2 + \omega^2}}$, $\sin\delta = \frac{\omega}{\sqrt{k^2 + \omega^2}}$.

For our case, steady state solution is

$$S(t) = 15 - 7\cos(\frac{\pi}{12}t - \delta) = 15 - 7\cos\frac{\pi}{12}(t - 3.5)$$

$$\tan\delta = \frac{\omega}{k} = \frac{\pi/12}{0.2} = \frac{5\pi}{12} \quad (\delta \approx 52.6^\circ)$$

Amplitude of our solution is $\frac{k}{\sqrt{k^2 + \omega^2}} T_1 = \frac{1}{\sqrt{1 + (\omega/k)^2}} T_1 \approx (1 - 0.5(\omega/k)^2) T_1$

The less k is, the less the change of temperature inside building is.

In other words, k measures sensitivity of building inside to outside temperature change.