Module Integration by parts

Technique of integration: integration by parts

1. Integration by parts

Sometimes, we need to integrate functions of mixed type such as $e^x \cos x$, $x^2 \sin x$, $x^3 \ln x$.

the derivative of the product of two given functions:

$$\frac{d}{dx} \ u(x)v(x) = u'v + uv'$$

Integrate both sides, then

$$\int \frac{d}{dx} u(x)v(x)\,dx = \int u'v + uv'\,dx$$

which yields

$$u(x)v(x) = \int u'v\,dx + \int uv'\,dx,$$

equivalently

$$\int uv'\,dx = u(x)v(x) - \int u'v\,dx.$$

This is called <u>integration by parts.</u>

Ex) Evaluate $\int x \cos x\,dx$

$$\int x(\sin x)'\,dx = x\sin x - \int (x)'\sin x\,dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C$$

How to use integration by parts: Choose u and v'

$u = x$	$v' = \cos x$	$\downarrow$	Simpler
$u' = 1$	$v = \sin x$		

$u = \cos x \qquad v' = x \qquad \downarrow$ more complicated (this choice is not good!!)

$u' = -\sin x \qquad v = \frac{1}{2}x^2$

You need to check whether your choice of u and v' is right or not.

Ex) Evaluate $\int x e^x dx$

① $u = x \qquad v' = e^x \qquad \downarrow \qquad \int e^x dx$ is simpler than original integral

$u' = 1 \qquad v = e^x$

② $u = e^x \qquad v' = x \qquad \downarrow$

$u' = e^x \qquad v = \frac{1}{2}x^2$

$\int \frac{1}{2}x^2 e^x dx$ is more complicated !! (does not work!!)

① $\int u v' dx = uv - \int u' v dx$

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx$$

$$= x e^x - e^x + C$$

Exercise) Evaluate $\int x e^{3x} dx.$

Example) Evaluate $\int \frac{\ln x}{x^2} dx$

① $u = \frac{1}{x^2} \qquad v' = \ln x$

$u' = -\frac{2}{x^2} \quad v = ?$

② $u = \ln x \quad v' = x^{-2}$

$u' = \frac{1}{x} \quad v = -x^{-1}$

$$\int (\ln x)(x^{-2})\,dx = (\ln x)(-x^{-1}) - \int \frac{1}{x}\left(-\frac{1}{x}\right)dx$$
$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$
$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

Example) (Case the function chosen to be integrated is implicit)

Evaluate $\int \ln x\,dx$ (lnx = 1 times lnx)

$u = \ln x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$\int (\ln x)(1)\,dx = (\ln x)(x) - \int \frac{1}{x}\,x\,dx$ = x ln x - x + C

Example) (Case of applying integration by parts twice)

Evaluate $\int x^2 e^x\,dx$

$u = x^2 \quad v' = e^x$

$u' = 2x \quad v = e^x$

$I = x^2 e^x - \int \underline{2xe^x}\,dx$

In general $I_n = \int x^n e^x\,dx$

<u>**Reduction formula**</u> $I_n = x^n e^x - \int n x^{n-1} e^x\,dx = x^n e^x - n I_{n-1}$

Exercise) Use reduction formula to evaluate $\int x^3 e^x\,dx$.

Need three step-reduction to arrive integral of e^x.
