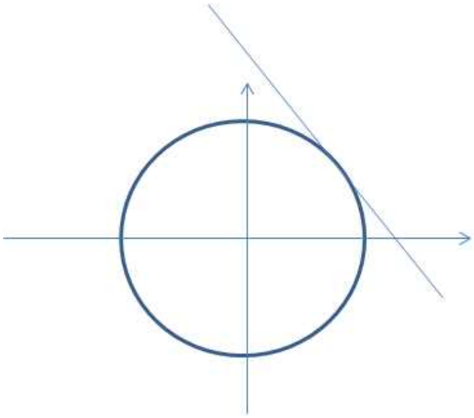
# Module Implicit differentiation

## 1. Motivating Problem and description of method

How can you find the slope of the tangent line to given circle



What we can use is an equation of circle : center (0,0) and radius =2

=> $x^2 + y^2 = 4$

It is not graph of function.

We take a part of circle such that it can be considered as the graph of a function.

=> Take upper semi circle

=> graph of function $y = f(x)$

=> $x^2 + f(x)^2 = 4$. Solve for f(x) => $f(x) = \pm\sqrt{4 - x^2}$

Since f(x) >0, $f(x) = \sqrt{4 - x^2}$ => slope is derivative of f(x)

BUT, if we want to get only derivative, say $\frac{dy}{dx}$ at given point, we don't need to know $f(x)$.

**(IDEA)**

Consider y as a dependent variable of x. That is $y = f(x)$. But it is defined implicitly not explicitly. (Can you understand its meaning?)

$x^2 + y^2 = 4$

Whenever you choose a value for x, you can determine y uniquely once you restrict range y varies. (=> Function is defined)

Set $y \geq 0$. If x=0 is chosen, then $(0)^2 + y^2 = 4 \Rightarrow y = 2$.

If x=1 is chosen, then $(1)^2 + y^2 = 4 \Rightarrow y = \sqrt{3}$.

Equation $x^2 + y^2 = 4$ gives a unique relation for a pair (x, y) if a restriction on y is imposed. That relation is a rule of assigning a number (y) on each number (x). That is a function. (Implicitly defined function)

=> Derivative $\frac{dy}{dx} = ?$

Apply $\frac{d}{dx}$ to $x^2 + y^2 = 4$.

=> $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}4$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

*Here $y^2$ is a composite function of x. Thus need to apply chain rule to find $\frac{d}{dx}y^2$ Solve for $\frac{dy}{dx}$. Then $\frac{dy}{dx} = -\frac{x}{y}$

=> If we take a point (a, b) on $x^2 + y^2 = 4$, then $\frac{dy}{dx}|_{(x,y)=(a,b)} = -\frac{a}{b}$

Derivative of implicit function is determined at given point.

=> Such a method is called **"implicit differentiation"**.

**Example)** Slope of tangent line at $(1, \sqrt{3})$ = $-\frac{1}{\sqrt{3}}$

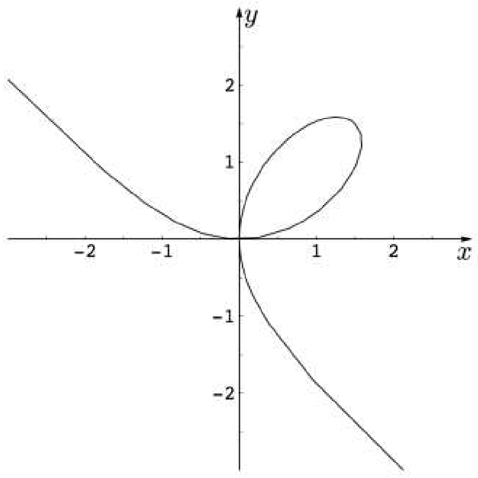
**Example)** (Folium of Descartes) $x^3 + y^3 = 6xy$. Find the slope of the tangent lin at (3,3).

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}6xy$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6(y + x\frac{dy}{dx})$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$




$$\frac{dy}{dx}\Big|_{(x,y)=(3,3)} = \frac{2(3)-3^2}{3^2-2(3)} = -1$$

Using that, we can find where the tangent line is horizontal.

$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x} = 0$ => $2y-x^2=0$ and the point (x, y) is on $x^3+y^3=6xy$.

=> $x^3+(x^2/2)^3 = 6x(x^2/2)$

$$x^3+\frac{1}{8}x^6 = 3x^3$$

$$x^3=0 \text{ or } x^3=16$$

=> (x, y)=(0, 0)

$x=\sqrt[3]{16}$, $y=\frac{1}{2}(\sqrt[3]{16})^2=\sqrt[3]{32}$.

## 2. Applications

- Derivation of power rule (rational exponent case) using implicit differentiation
- Example: annual daylight hours of Seoul (Application of chain rule)

### 1) Derivation of power rule using implicit differentiation

(positive integer/ negative integer case were proved)

$y=x^{p/q}$ (p, q: nonzero integers, no common factor)

Power rule $\frac{dy}{dx} = \frac{p}{q}x^{p/q-1}$

**Example)** $\frac{d}{dx}x^{-2/3} = -\frac{2}{3}x^{-2/3-1} = -\frac{2}{3}x^{-5/3}$

**(1) Proof of power rule (rational exponent case)**

We assume that power rule holds for integer exponent case.

$y = x^{p/q}$ => $y^q = x^p$

$\frac{d}{dx}y^q = \frac{d}{dx}x^p$ => $qy^{q-1}\frac{dy}{dx} = px^{p-1}$

$\frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}(x^{p/q})^{1-q} = \frac{p}{q}x^{p-1+p/q-p} = \frac{p}{q}x^{p/q-1}$

(Here we applied implicit differentiation.)

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**2) Example of annual daylight hours of Seoul**

In example of daylight hours of Seoul, when is the length of daylight hours most rapidly increasing (rep. decreasing)? How rapidly is the daylight hours increasing on that day?

$y = 12 + 2.7\sin\frac{2\pi}{365}(t-88)$

y: daylight hours after t days from January 1st.

Rate of change = $\frac{dy}{dt} = \frac{d}{dt}(12 + 2.7\sin\frac{2\pi}{365}(t-88)) = 0 + 2.7\frac{d}{dt}sin\frac{2\pi}{365}(t-88)$

$= 2.7\cos\frac{2\pi}{365}(t-88)\frac{d}{dt}(\frac{2\pi}{365}(t-88))$

$= 2.7 \times \frac{2\pi}{365}cos\frac{2\pi}{365}(t-88)$

$\frac{dy}{dt}$ has biggest value where cosine has value 1, that is,

$2\frac{\pi}{365}(t-88) = 0,\ 2\pi,\ 4\pi, \ldots$

$t = 88,\ 88 + 365, \ldots$

=> Rate of change is biggest at 88$^{th}$ day (=> Daylight hour is most rapidly increasing at 88th day)

What is the rate of change then?

$\frac{dy}{dt} = 2.7 \times \frac{2\pi}{365} = 0.04$ hours = $0.04 \times 60 = 2.4$ minutes

=> At 88th day, daylight hours increase at a rate of 2.4 min/day