Module Finding extreme values (Application of derivative)

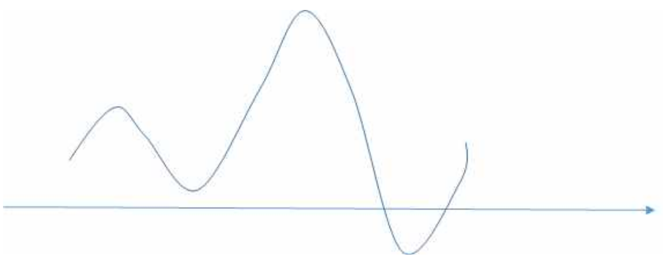
- Local max/min
- Fermat's theorem
- Critical numbers
- Closed interval method
- First derivative test

1. Motivation: Optimization problem

Example) The concentration C(t) (in mg/cm^3) of a drug in a patient's blood stream after t hours is $C(t) = \dfrac{0.016t}{t^2+4t+4}$. When does the concentration reach the highest value?

General question: Find the largest and smallest values of given function.

Definition) A function f is defined on [a, b] (or more general interval).
(i) The point (c, f(c)) is the highest point on the graph, that is, $f(c) \geq f(x)$ for all x in [a, b]. Then f is said to have absolute maximum at x=c and f(c) is called the absolute maximum value of f on [a, b].
(ii) The point (d, f(d)) is the lowest point on the graph, that is, $f(d) \leq f(x)$ for all x in [a, b]. Then f is said to have absolute minimum at x=d and f(d) is called the absolute minimum value of f on [a, b].



Example) $f(x) = \sin x$ over $[0, 2\pi]$. f has absolute max at $x = \pi/2$. abs max value is 1. f has absolute minimum at x=$3\pi/2$. abs min value is -1

Question

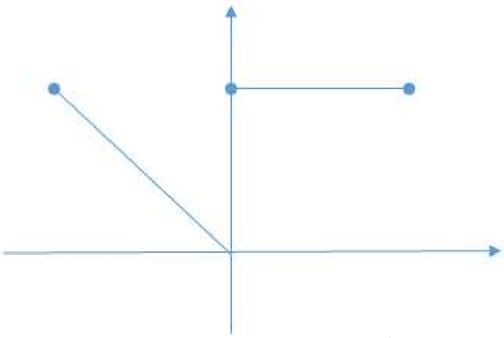
Does a function always have absolute maximum or minimum?

Example) f(x)=x on R (Domain is unbounded)

Example) f(x) = x on [-1, 1].

Example) $f(x)=x^2$ on (-1, 1) (Domain is missing boundary points (end points of the interval) => f does not have abs max.

Example) $f(x)=\begin{cases} -x, & -1lex<0 \\ 1, & 0\le x\le 1 \end{cases}$ Domain= [-1, 1], bounded and closed.



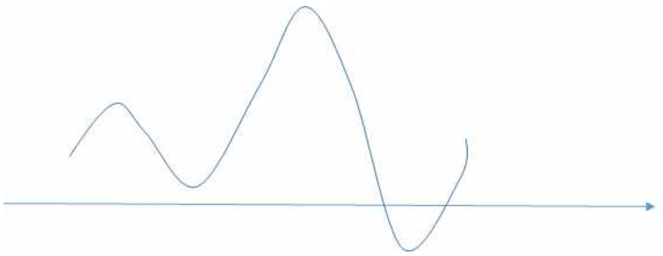
=> f is not continuous at x=0

Observation) Condition on the interval and continuity are required.

2. Extreme value theorem

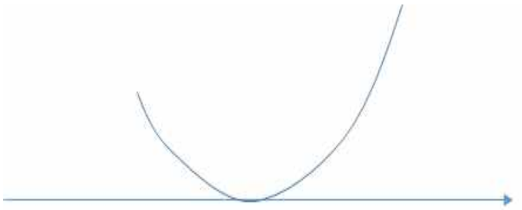
If f is <u>continuous</u> on a <u>closed bounded</u> interval [a, b], then f has absolute maximum and absolute minimum somewhere in [a, b].

Question: A continuous function defined on a bounded closed interval has both absolute max and absolute min. How can we find these values (locations)?



Definition) The function f is said to have **a local maximum (resp. minimum)** at $x=a$ if there exists an interval (α, β) including $x=a$ such that f has the greatest (resp. least) value at $x=a$ on (α, β).

Example) $f(x)=x^2$ *on* $[-1,2]$. f has absolute minimum at x=0, which is also local minimum.



Question) How can we find local max/min? Characterize these local extreme points.

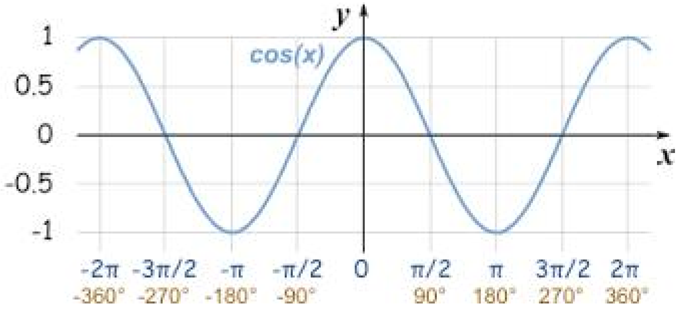
Observation) The tangent line at local max/min points is horizontal, that is, if f is differentiable, its derivative vanishes here.

Fermat Theorem Suppose f has a local maximum or a local minimum at the point $x=a$. Suppose that f is differentiable at $x=a$. Then $f'(a)=0$.

Caution) Its converse is not true. $f'(a)=0$ does not imply that f has either local max or local min at x=a.

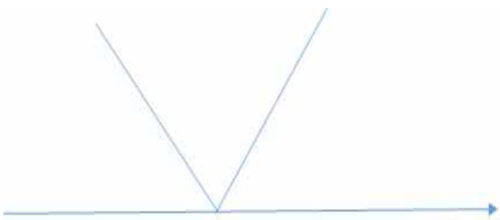
Example) $f(x)=x^3$. $f'(0)=0$, but x=0 is neither local max nor local min.

Example) The function $f(x) = \cos x$ is defined on $(-\pi, 3\pi)$. Then f has a local maximum at $x=0$ and $x=2\pi$. It holds that $f'(0) = -\sin(0)$ and $f'(2\pi) = -\sin(2\pi) = 0$. At $x=\pi$, f has a local minimum. It holds that $f'(\pi) = -\sin(\pi) = 0$.



Question) What if f is not differentiable where f has local max or local min?

Example) Consider the function $g(x) = |x|$. It has a local minimum at x=0, but it is not differentiable at x=0.



Definition) The function f is defined on the interval (a, b). The point $x=c$, which is between a and b, is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ does not exist.

Example) Find the critical numbers of $f(x) = (x^2 - x)^{2/3}$.

$f'(x) = (2/3)(x^2-x)^{2/3-1}(2x-1) = (2/3)\dfrac{2x-1}{\sqrt[3]{x^2-x}}$. (Apply chain rule)

$f'(x) = 0 \Leftrightarrow$ 2x-1=0 => The derivative of f vanishes at $x = 1/2$.

$f'(x)$ is not defined at $x = 0,\ 1$. (denominator vanishes <=> $x^2 - x = 0$)

Thus f has critical points at $x = 1/2, 0, 1$.

Exercise) Find the critical numbers of $g(x) = x^{3/5}(4-x)$.

Remark) Critical numbers of function are interior points of the interval. Boundary points (End points) are not endpoints.

Example) $h(x) = \sqrt{4-x^2}$ is defined on $-2 \le x \le 2$

Its derivative $h'(x) = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4-x^2}}$ is defined only on (-2, 2).

Even if $\lim_{x\to 2} h'(x) = -\infty$, $\lim_{x\to -2} h'(x) = +\infty$, x=2 and x=-2 are not considered critical numbers. (In beginning, we not consider the derivative of given function at two boundary points because derivative itself is considered at only interior point)

=> We have only to search for critical numbers insider the interval.

*Critical numbers are candidates for local max/local min (Fermat theorem)

----

3. "Closed interval method"

Finding <u>absolute max and min of continuous function</u> f defined on a bounded closed interval [a, b].

1. Find critical numbers of f in (a, b).
2. Find the values of f at critical numbers.
3. Find the values of f at the endpoints of [a, b], that is, f(a) and f(b).
4. The largest value is abs max and the smallest value is abs min.

Example) Find the extreme values of $f(x) = (x^2-x)^{2/3}$ over $-1 \le x \le 2$.

step 1. critical numbers : critical numbers are -1/2, 0, 1

step 2. f(-1/2), f(0), f(1)

step 3. f(-1), f(2).

Compare function values there.

$$f(0) = f(1) = 0,\ f(1/2) = 1/\sqrt[3]{16},\ f(-1) = f(2) = \sqrt[3]{4}.$$

Thus f has an abs maximum at -1 and 2 and has an abs minimum at 0 and 1.

Exercise) (Strongly recommend them)

Find abs max and abs min of following function on given interval using closed interval method.

(1) $g(x) = x^3 - 3x^2 + 1, \quad -0.5 \le x \le 4$

(2) $h(x) = \dfrac{x}{x^2 - x + 1}, \quad [0,3]$

(3) $f(x) = x\sqrt{9 - x^2}, \quad [-2,3]$

(4) $k(x) = \sqrt[3]{x}(8 - x), \quad [0,8]$

Question) What are the extreme values of $f(x) = (x^2 - x)^{2/3}$ over entire real line?

We can compare function values of several local maximum points. But the biggest value of them may not be the absolute max. (abs max may not exist in open interval)

=> We need to know entire behavior of function

Characterization of local max and local min using derivative

4. First derivative test.

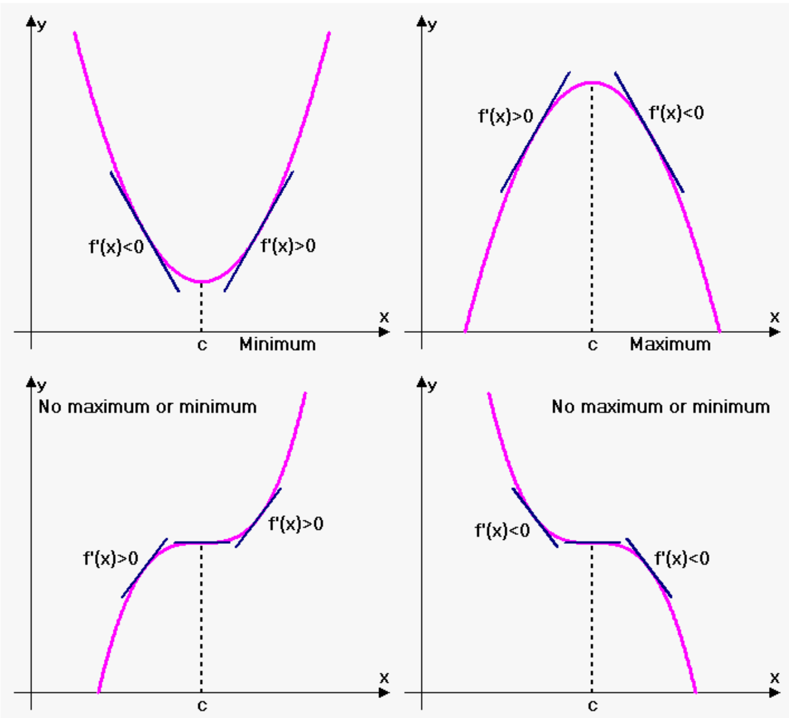
Suppose that f has a critical point at $x = c$ and f is differentiable around $x = c$. (f'(c) may not exist)

(1) If the sign of f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$.

(2) If the sign of f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$.

Example) Consider $f(x) = (x^2 - x)^{2/3}$ over the entire real line. It has critical points at $x = 1/2,\ 0,\ 1$.

For $x < 0$, $\quad f'(x) = (2/3)\dfrac{2x-1}{\sqrt[3]{x^2 - x}} < 0 \quad (x^2 - x = x(x-1) > 0$ for $x < 0)$



and

for $0 < x < 1/2$, $f'(x) > 0$ $(x^2 - x = x(x-1) < 0$ for $0 < x < 1$.

=> f has a local minimum at $x = 0$.

For $1/2 < x < 1$, $f'(x) < 0$. Thus f has a local maximum at $x = 1/2$.
sign is positive => sign is negative
(sign changes from plus to negative)