Module Exponential model

- **Exponential model**
- Additional example of exponential growth model

1. Exponential model (Stewart section 6.5) (★★★★★)

(1) Population growth model

Example. The population of Mexico in the early 1980s is given as follows.

Year	Population (millions)	Change in population (millions)
1980	67.38	
1981	69.13	1.75
1982	70.93	1.80
1983	72.77	1.84
1984	74.66	1.89
1985	76.60	1.94
1986	78.59	1.99

Ratio of each year's population to the previous year's population.
(Population in 1981)/(Population in 1980)= 69.13/67.38 =1.026
(Population in 1982)/(Population in 1981)= 70.93/69.13 =1.026

Population t years after 1980 is given by $P = 67.38(1.026)^t$.

$$\frac{P(t+1)}{P(t)} = C = 1 + k \quad \Rightarrow \quad \frac{P(t+1) - P(t)}{P(t)} = C - 1 = k$$

$$\Rightarrow \frac{\Delta P}{P} = k \quad \Rightarrow \quad \frac{\Delta P}{\Delta t} = kP \quad \Rightarrow \quad \frac{dP}{dt} = kP$$

Consider differential equation $\frac{dy}{dt} = ky$ (y= y(t))where k is a constant. If k is positive, it gives a model for population growth. Since its solution is increasing exponential function, it is called <u>**exponential growth model.**</u>

Constant k is called <u>relative growth</u> since $\frac{dP}{dt}/P = k$

If k is negative, then its solution is decreasing exponential function, thus the differential equation gives exponential decay model.

(How to find solution)

Since it is separable, it is solved by separation as follows:

$$\frac{1}{y}dy = kdt$$

then integrate both sides,

$$\int \frac{1}{y}dy = \int kdt$$

$$\ln|y| = kt + C$$

Solve for y. We have $|y| = e^{kt+C}$ and $y = \pm e^C e^{kt}$.

Let $A = \pm e^C$ be a new constant.

Then solution is $y = Ae^{kt}$.

The constant A is determined by $y(0) = Ae^0 = A$.

Thus the solution is $y(t) = y(0)e^{kt}$.

EXAMPLE (Population growth)

Population of India

Year	Population (millions)
1951	361
1961	439
1971	548
1981	683
1991	846
2001	1029

If we set the population in 1951 as initial population, that is, P(0)=361 and use P(10) to decide relative growth rate k, $P(t) = P(0)e^{kt}$ and $P(10) = P(0)e^{10k}$. Then we have

$$e^{10K} = \frac{439}{361}$$

$$10K = \ln\frac{439}{361}$$

$$K = \frac{1}{10}\ln\frac{439}{361} \fallingdotseq 0.019 = 1.9\% \text{ (annual relative growth)}$$

Check whether our model fits with real data. We have

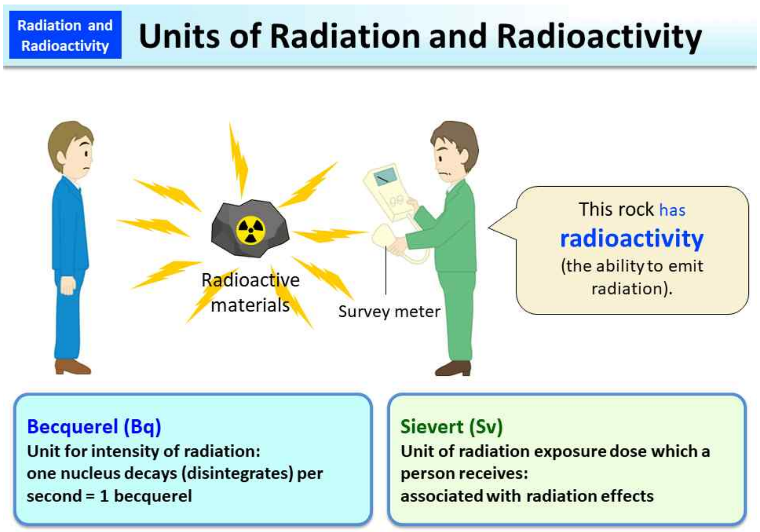
$$P(20) = P(0)e^{20K}$$
$$= P(0)e^{2\ln\frac{439}{361}}$$
$$= P(0)\left(\frac{439}{361}\right)^2$$
$$\fallingdotseq 361 \times 1.4788 \fallingdotseq 533 \quad \text{(real population 548)}$$

and

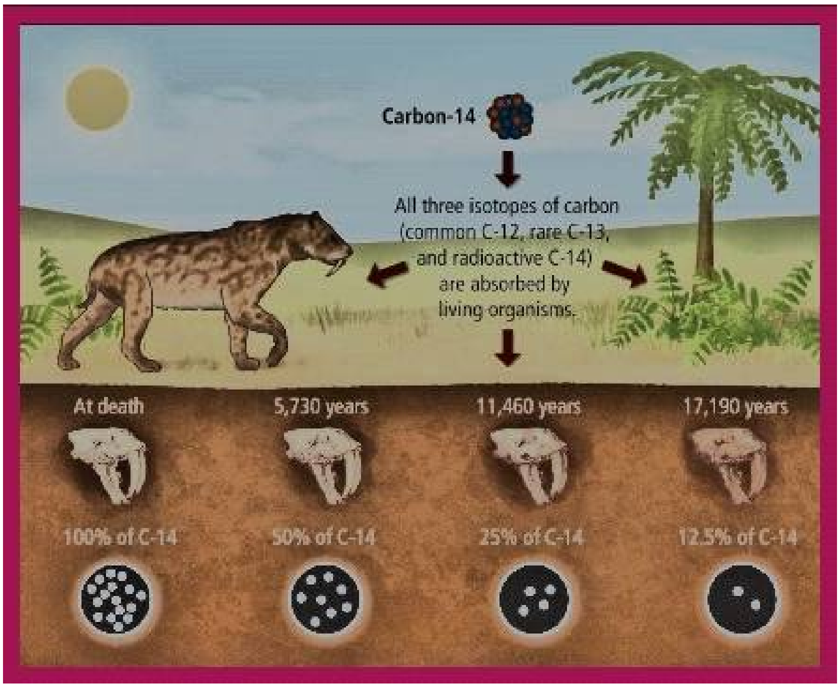
$$P(30) = P(0) \times e^{30K}$$
$$= P(0) \times e^{3\ln\frac{439}{361}}$$
$$= P(0) \times \left(\frac{439}{361}\right)^3$$
$$\fallingdotseq 361 \times 1.7983 \fallingdotseq 649 \quad \text{(real population 683)}$$

They are much less than the real data.

Example. (Exponential decay) Scientists have observed that if a quantity R_0 of a radioactive material is present at time t=0, then the quantity present at time t is $R(t) = R_0 e^{-kt}$ where k is positive. ($\frac{dR}{dt} = -kR$)



Half-life of this material is defined to be the time it takes for the quantity to decrease by a factor of 1/2. From $R_0 e^{-kt} = \frac{1}{2}R_0$, half-life T is $\frac{\ln 2}{k}$.



carbon dating (application of radioactive decay-exponential model)

Exercise. It is known that Radon-222 has a half-life of 3.825 days. Determine how long will it take for 80% of the Radon-222 to decay.

(2) Newton's law of cooling

Another model using exponential function is found in describing cooling or heating of an object. Newton's law of cooling says that

the rate of cooling of an object
$\propto$ temperature difference between
the object and its surrounding

where the rate of cooling =the rate of decrease in temperature of the object with respect to time.

Suppose that an very hot object is placed and its surrounding is under much lower temperature, then

T=T(t) : temperature of hot object after t (minutes or hours) from initial moment

$$\frac{dT}{dt} = k(T - T_s)$$

where $T(t)$ = temperature of the object at time t,

T_s = temperature of the surroundings (=constant) and

k is a proportional constant.

In our case k is negative.

This is 1st order differential equation. How to solve it?

Set $y = T - T_s$, then ($\frac{dy}{dt} = \frac{dT}{dt}$ because T_s is constat)

$$\frac{dy}{dt} = ky$$

$$y = y(0)e^{kt}$$

$$T(t) = T_s + (T(0) - T_s)e^{kt}$$

Example. A bottle of cider at room temperature (22℃) is placed in a refrigerator where its temperature is 7°. After half an hour the cider has cooled to 16°. How long does it take for the cider to cool to 10℃?

(SOL) T(t)= temp of cider at t minutes after it is placed at refrigerator.

Since $T(0) = 22$, $Ts = 7$, we have

$$T(t) = 7 + (22-7)e^{Kt}$$

(i) We can decide k using $T(30) = 7 + 15e^{30K} = 16$. Then

$$e^{30K} = \frac{16-7}{15} = \frac{9}{15}$$

$$30K = \ln\frac{9}{15}$$

$$K = \frac{1}{30}\ln\frac{9}{15}$$

$$= -\frac{1}{30}\ln\frac{5}{3}$$

(ii) To decide when the temperature of the cider becomes 10, set

$$T(t) = 7 + 15e^{Kt} = 10$$

solve for t

$$e^{Kt} = \frac{3}{15} = \frac{1}{5}$$

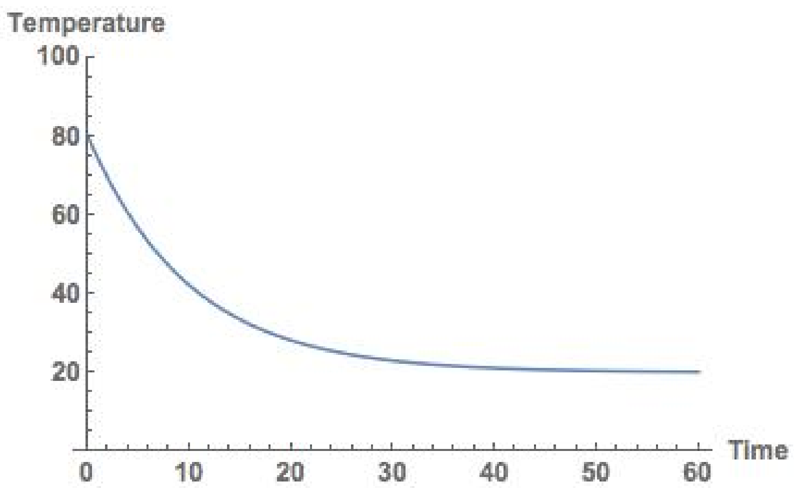
$$Kt = \ln\frac{1}{5}$$

$$t = -\frac{1}{K}\ln 5 = \frac{\ln 5}{\frac{1}{30}\ln\frac{5}{3}}$$

$$= 30\frac{\ln 5}{\ln 5 - \ln 3}$$

$$= \frac{30}{1-\frac{\ln 3}{\ln 5}} \risingdotseq 30 \times 3.15 \risingdotseq 94$$

where $\frac{\ln 5}{\ln 5 - \ln 3} = \frac{1.6094}{1.6094 - 1.0986} = \frac{1.6094}{0.51-08} = 3.15$. The cider has a temperature 10 degree at 94 minutes after we put the cider at refrigerator.



2. Modeling of spread of contagious disease

Daniel Bernoulli (1760) : small pox (SI model / Today SIR model is used)

x=portion of susceptible

y= portion of infectious (=those who have disease and can infect others)

x+y =1

Assumption: spread by contact

$\frac{dy}{dt} \propto$ number of contacts between susceptible and infectious = xy

=> $\frac{dy}{dt} = kxy = ky(1-y)$ (=proportional constant, positive)

y(0)=y_0 (initial portion of infectious)

=> want to find out unknown function y= y(t)

$$\frac{dy}{y(1-y)} = kdt$$

$$\int \frac{1}{y(1-y)}\, dy = \int k\, dt$$

$$\int \frac{1}{y} + \frac{1}{1-y} dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right| = kt + C$$

$$\frac{y}{1-y} = \pm e^{kt+C} = Ae^{kt}$$

(Here $\frac{d}{dy} ln|1-y| = (\ln|\cdot|)' \frac{d}{dy}(1-y) = \frac{1}{1-y}(-1) = -\frac{1}{1-y}$ chain rule is used)

(Here set A= $\pm e^C$)

$$\frac{1-y}{y} = \frac{1}{A} e^{-kt}$$

$$\frac{1}{y} = 1 + \frac{1}{A} e^{-kt}$$

$$y = \frac{1}{1+Be^{-kt}}$$

where we set B=1/A

Here y(0)= 1/(1+B) is between 0 and 1 => B is positive constant

=> our function is composition of two decreasing function

=> our function is increasing (You can check that dy/dt > 0)