Module Derivative: definition

Today's topics

- Velocity Problem (motivation)
- Tangent Problem
- Definition of Derivative

1. Velocity Problem

Want to describe a motion of a moving particle on a straight line.

|------ s=f(t) -------->

A position function $s = f(t)$, $t \geq 0$ where t represents a time

(where s is measured in meters and t is measured in seconds)

t (sec)	0	1	2	3	4	5
s (meters)	0	3	7	8	6	4

Can you describe the motion by looking at the table?

Want to know how fast the particle moves.

DEF) <u>Average velocity over time interval</u> $a \leq t \leq b$ as follows

$$\text{average velocity} = \frac{\text{change of position}}{\text{time elapsed}} = \frac{f(b)-f(a)}{b-a}$$

Example) Given position function is $s = f(t) = t^2$.

Average velocity over $1 \leq t \leq 2$: $\frac{2^2-1^2}{2-1} = 3\,m/s$

Average velocity over $2 \leq t \leq 3$: $\frac{3^2-2^2}{3-2} = 5\,m/s$.

What does it tell us?

Question) Can we find out how fast the particle is moving at a given moment?

=> For instance, if we want to define and find the velocity at $t=1$, we can consider average velocity over very short time interval beginning at $t=1$. Let's keep taking shorter and shorter time interval starting at t=1 as follows.

Time interval	Average velocity
$1 \le t \le 1.2$	$\frac{(1.2)^2-1^2}{1.2-1}=2.2\ m/s$
$1 \le t \le 1.1$	$\frac{(1.1)^2-1^2}{1.1-1}=2.1\ m/s$
$1 \le t \le 1.01$	$\frac{(1.01)^2-1^2}{1.01-1}=2.01\ m/s$
$1 \le t \le 1.001$	$\frac{(1.001)^2-1^2}{1.001-1}=2.001\ m/s$

From the table, we can find a certain tendency of average velocities. They seem to converge to 2.

=> Instantaneous velocity at t=1
=> a limit of average velocities over time intervals beginning at t=1 as time intervals become shorter and shorter.

=> $$v(1)=\lim_{t\to 1}\frac{f(t)-f(1)}{t-1}=\lim_{t\to 1}\frac{t^2-1^2}{t-1}.$$

Definition) For a given position function $s=f(t)$, the velocity (or instantaneous velocity) at t=a is defined by

$$v(a)=\lim_{t\to a}\frac{f(t)-f(a)}{t-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

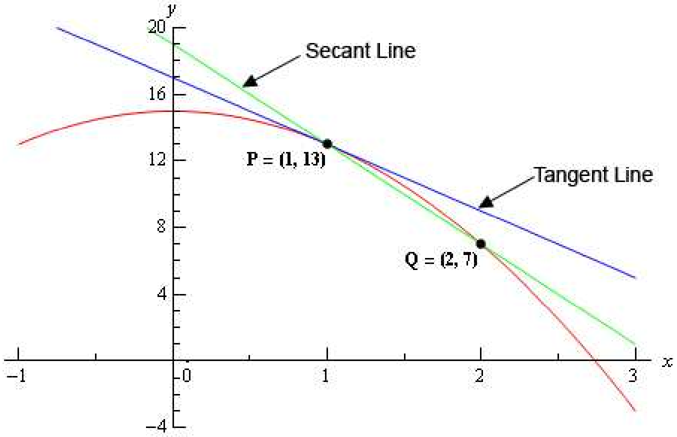
Exercise) For given position function below, find the velocity at given time.

(a) $s=\frac{2}{t+1}$, $t=1$

(b) $s=\sqrt{t}$, $t=4$

2. Tangent Problem

Classic problem of finding the slope of the tangent line to a smooth curve at a point



Question: How can we define and find the slope of the tangent line (blue one) to curve (red one)?

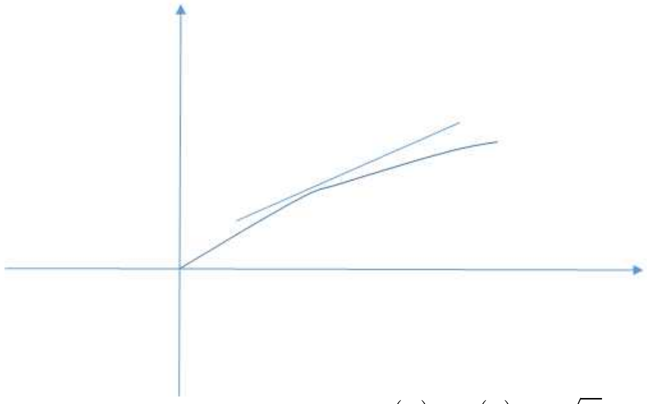
Idea: To define the slope of the line, we need two distinct points on the line.
Take a point Q near P and consider secant line (green line) passing through P and Q.
Find the slope of the line PQ.
Green line and blue line slightly different.
Observation) As the point Q approaches the point P along the curve, the secant line (green one) rotates and approaches to the tangent line (blue one).
=> Limiting process
=> If a limit exits, then that limit is good candidate for the slope of the tangent line.
=> If a limit does not exists, then definite tangent line is not determined. The curve does not allow tangent line at that point.

Slope of the secant line PQ

$P(a,f(a))$, $Q(x,f(x))$ => slope = $\frac{f(x)-f(a)}{x-a}$

=> slope of the tangent line $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Example) Find the slope of then tangent line to curve $y = \sqrt{x}$ at $(2, \sqrt{2})$.



Slope of the secant line = $\frac{f(x) - f(2)}{x - 2} = \frac{\sqrt{x} - \sqrt{2}}{x - 2}$

Slope of the tangent line = $\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$

Q) Why can't we apply DSP?

$$\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} = \lim_{x \to 2} \frac{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})}{(x-2)(\sqrt{x} + \sqrt{2})}$$

$$= \lim_{x \to 2} \frac{x - 2}{(x-2)(\sqrt{x} + \sqrt{2})}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Example) Find the slope of the tangent line to curve y=$f(x) = x^2$ at $(3,9)$.

$$\lim_{x \to 3} \frac{x^2 - 3^2}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} x + 3 = 6$$

Derivative: definition and computation

Definition) The derivative of $f(x)$ at $x = a$ is defined as a number, denoted by $f'(a)$ such that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

3. Meaning of derivative $f'(a)$

(1) Instantaneous rate of change

Derivative $f'(a)$ measures rate of change of f at x=a. It measures how rapidly $f(x)$ changes at x=a. In ordinary terms, it tells us growth rate if f represents a certain quantity depending on time.

Example) $f(x)=3x+1$

$f'(a)$=3 at each point a. Deraivative is constant. It means that Rate of Change of this function is constant number 3 every where. At each number a, the function grows (increases) at a rate of 3.

Example) $g(x)=x^2$

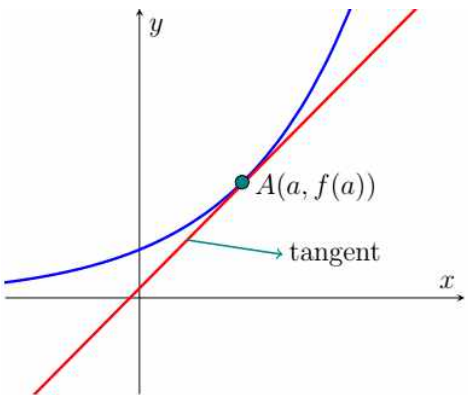
$g'(1)=2,\ g'(2)=4,\ g'(3)=6$

Derivatives of g are different at each point. It means that the function changes at different rate. The function increases at a rate of 2 at 1, increases at a rate of 4 at 2. It means that the function increases more rapidly at x=2 than at x=1.

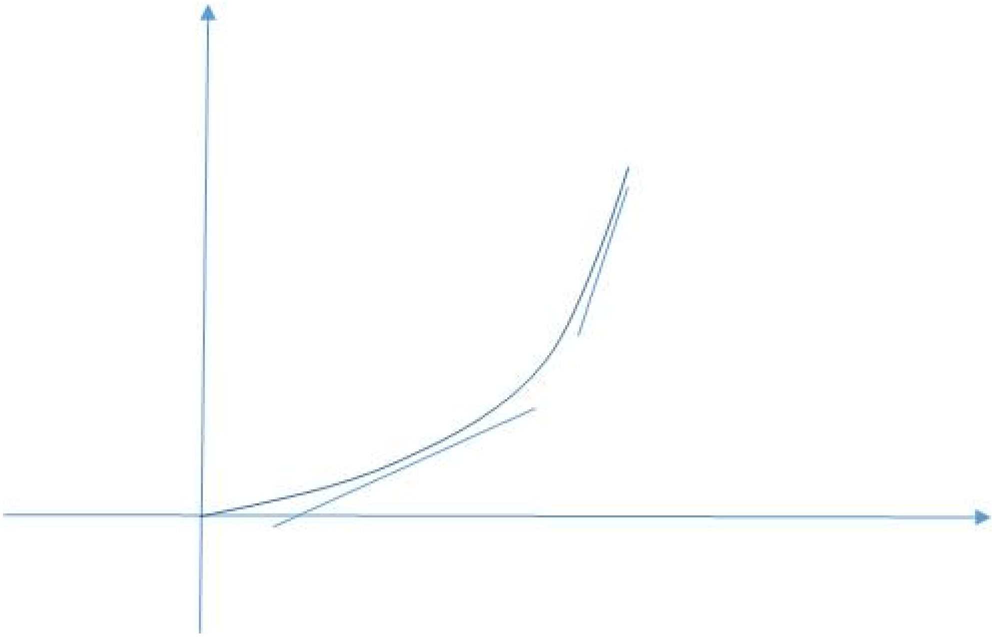
(2) Geometric meaning

Derivative $f'(a)$ represents the slope of the tangent line to $y=f(x)$ at $(a,f(a))$.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$



Tangent line at $(a,f(a))$ visualizes the rate of growth of the function at x=a. If the tangent lin is steep, it means that the function changes rapidly at that point.



The steeper tangent line is, the more rapidly the function increases.

4. How to compute using definition?

Example) $f(x) = \frac{1}{x^2}$. Find $f'(2)$.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{x^2} - \frac{1}{2^2}}{x - 2}$$

$$= \lim_{x \to 2} \frac{4 - x^2}{4x^2(x-2)} = \lim_{x \to 2} \frac{(2-x)(2+x)}{4x^2(x-2)} = \lim_{x \to 2} \frac{-(2+x)}{4x^2} = -\frac{4}{16} = -\frac{1}{4}$$

Computational Tip: Reparametrization variable x which approaches the point a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{==>} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

where we set $x = a + h$, then as h -> 0, x -> a.

(It is useful when the function is polynomial or rational function)

Example) $g(x) = x^3$. Find $g'(2)$.

$$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h}$$

$$= \lim_{h \to 0} \frac{8 + 3 \times 4h + 3 \times 2h^2 + h^3 - 8}{h}$$

$$= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \to 0} 12 + 6h + h^2 = 12$$

*Recall $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Ref: Stewart section 1.4 and 2.1

Strang Calculus (MIT open course-ware) section 1.3, section 2.1 and 2.3