Module Derivative: Basic rules

1. Rate of change (=meaning of derivative)

x: independent variable

y: dependent variable

$y = f(x)$: y changes depending on x and that change is determined by f

If x changes from x_1 to x_2, then x changes by $x_2 - x_1$, which is called increment of x, denoted with Δx

The corresponding change in y is $f(x_2) - f(x_1)$, denoted with Δy

Interested in $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, which is called average rate of change of y with respect to x

Example) If x=time, t and y=position, s, then $s = f(t)$ is position function of a moving particle. For this case, average rate of change of s with respect to t, which is $\frac{\Delta s}{\Delta t}$, is in fact average velocity.

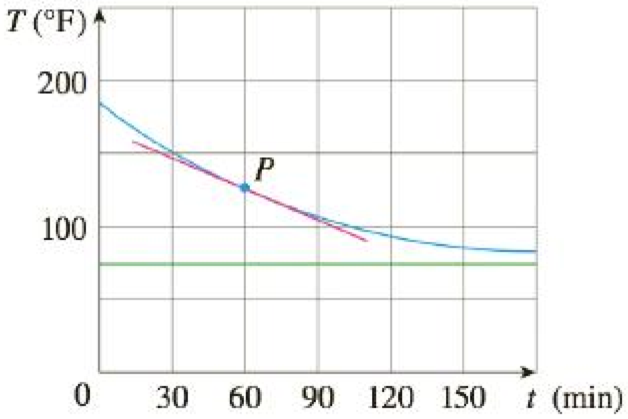
Derivative of f at a can be understood as limit of average rate of change

=> $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$

called instantaneous rate of change of f at a

$f'(a)$= **instantaneous rate of change of $y = f(x)$ with respect to x when x=a**

Example) A roast chicken taken from the oven is placed on the table. Its temperature is 185°F. The room temperature is 75°F. The temperature of the chicken depends (in fact decreases) on the time (measured in minutes). Describe the meaning of the derivatives of this function at several moments.



Let t be a time and u be a temperature of roast chicken. Then u changes depending on time. We have a function

$u = f(t)$

From the picture $f'(60) = \frac{110-145}{90-30} = -\frac{35}{60} = -7/12$=-0.58

What does it mean?

Rate of change of temperature with respect to time after 60 minutes passed from the moment the roast chicken is taken from oven.

=> Rate of decrease (Temperature is decreasing)

=> derivative is negative number

=> **It measures how rapidly the temperature decreases at the moment when 60 minutes passed from the beginning.**

=> Unit? Degree/minutes

=> rate of change = -0.58 degree/minute

=> $f'(60) \approx \frac{f(60+\Delta t) - f(60)}{\Delta t} \approx -0.58$

=> For instance if we know the temperature of chicken after 60 minute, we can predict its temperature after additional 2 minutes. Temperature decrease by 0.58 each minute

$f(60+\Delta t) \approx f(60) - 0.58 \times \Delta t$

$f(62) = f(60+2) \approx f(60) - 0.58 \times 2 = 125 - 1.16 = 123.84$

Exercise) Study the example 6, section 2.1 (derivative of cost function)

2. Derivative as a function

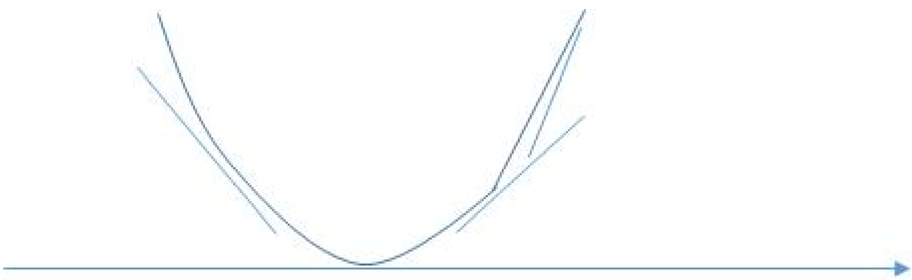
Any efficient way of computing derivative?

We can derive a new function f' from a given function f over an open interval I.

$I \ni a \to f'(a)$

$f' : x \to f'(x)$

Example



At each point of the curve, put tangent segment.

=> slope of the tangent segment = derivative of function.

x	$f'(x)$	x	$f'(x)$
-1	-1	0	0
-2	-2	1	1
		2	2

Derivative of a function $y = f(x)$ at x

=> $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Here we understand x as a fixed number for evaluating the limit.

Example)

$f(x) = x^2$ (from its graph we guess $f'(x)$ is a linear function)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} == \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

Exercise Find $g'(x)$ for $g(x) = \sqrt{x}$

Notation

$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ **(Leibniz Notation)**

Example) $f(x) = x^3$, $f'(x) = \frac{d}{dx}x^3$

3. Differentiation rule

Suppose that f and g are differentiable (means that f' and g' exist). Then

1. Derivative of constant function is 0.
2. $(cf)' = cf'$
3. $(f+g)' = f' + g'$
4. Power rule $\frac{d}{dx}x^n = nx^{n-1}$ (n: positive integer)
5. (Leibniz rule or product rule) $(fg)' = f'g + fg'$
6. (Quotient rule) $(f/g)' = (f'g - fg')/g^2$

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(1) Proof of power rule

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Use binomial expansion $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n$

Idea: Generalization of the case $y = x^2$

Implication of Rule 1~4 : Derivative of polynomials

Example) $\frac{d}{dx}(4x^3 - 5x^2 + 100) = (4x^3)' + (-5x^2)' + (100)' = 4(x^3)' - 5(x^2)' + (100)'$

$$= 4(3x^2) - 5(2x) + 0 = 12x^2 - 10x$$

Thus $f(x) = 4x^3 - 5x^2 + 100$, $f'(1) = (12x^2 - 10x)|_{x=1} = 12(1)^2 - 10(1) = 12 - 10 = 2$

Remark) Power rule can be extended to rational number

Example $\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$

(Power Rule) $\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{p/q-1}$

We can derive a formula using implicit differentiation (later)

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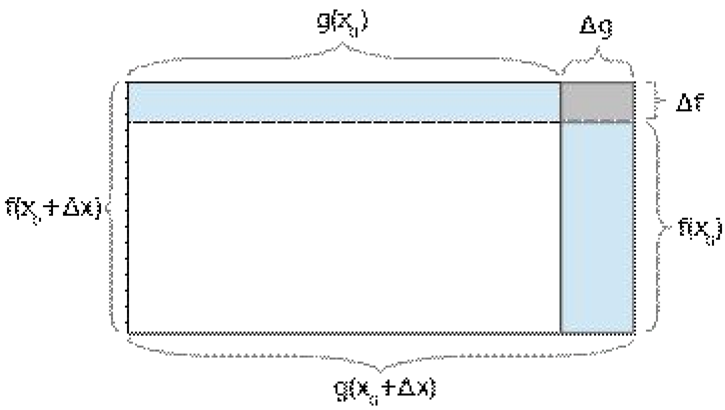
4. Leibniz Rule (Product Rule) $(fg)' = f'g + fg'$

Freshman's wrong formula

$(f \cdot g)' = f' \cdot g'$

(You can disprove with $f = g = x$)

We can not but differentiate one by one.



Rectangle with side length f(t) and g(t).

Rate of change of Area =?

=> $\frac{d}{dt}(f(t)g(t))$

$f(t+\Delta t)g(t+\Delta t) - f(t)g(t) = \Delta f(t) \times g(t) + f(t) \times \Delta g(t) + \Delta f(t) \times \Delta g(t)$

Example) $\frac{d}{dx}\sqrt{x}(x+1) = (x^{1/2})'(x+1) + \sqrt{x}(x+1)'$

$$= \frac{1}{2}x^{-1/2}(x+1) + \sqrt{x}(1) = \frac{x+1}{2\sqrt{x}} + \sqrt{x}$$

$$= \frac{x+1+2x}{2\sqrt{x}} = \frac{(3x+1)}{2\sqrt{x}}$$

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5. Quotient Rule

$(f/g)' = \frac{f'g - fg'}{g^2}$

Suppose that $1/g$ is differentiable, then apply product rule

$(f \cdot \frac{1}{g})' = f' \cdot \frac{1}{g} + f \cdot (\frac{1}{g})'$

Claim $(\frac{1}{g})' = -\frac{g'}{g^2}$

$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{h\, g(x+h) g(x)} \to -g'(x)/g(x)^2$ as h -> 0

(Here g(x+h) -> g(x) as h -> 0 <= g is continuous at x <= g is differentiable at x)

Back to $(f \cdot \frac{1}{g})' = f' \cdot \frac{1}{g} + f \cdot (\frac{1}{g})' = f'\frac{1}{g} + f(-\frac{g'}{g^2}) = \frac{f'g - fg'}{g^2}$

Example) $\frac{d}{dx}\frac{x+1}{x-1} = \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2}$

$$= \frac{(x-1) - (x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

Exercise) Derive a power rule for negative integer using quotient rule

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

Example) $\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx}x^{-2} = -2x^{-2-1} = -\frac{2}{x^3}$