Module Derivative of trigonometric functions

1. Modeling using trigonometric functions

(1) Operation on sine function (or cosine function)

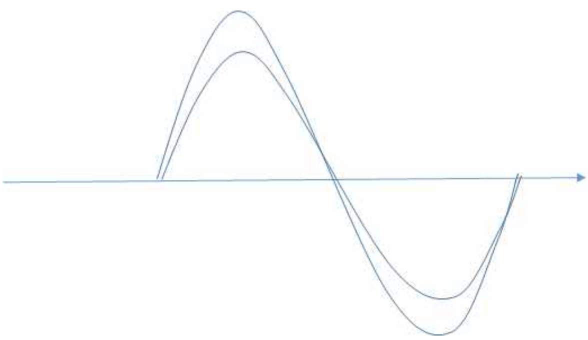
To express more general periodic function, we consider stretching and shifting of sine and cosine functions. General forms are as follows:

$$y = a + b\sin k(x-p), \quad y = a + b\cos k(x-p)$$

where sine and cosine functions are stretched to y-direction by b, x-direction by k, and moved to x-direction by p, y-direction by a.

=>The number b is called **amplitude** of the sine and cosine functions, it decides the largest and smallest function value.
=>The **period** for given k is $2\pi/k$.

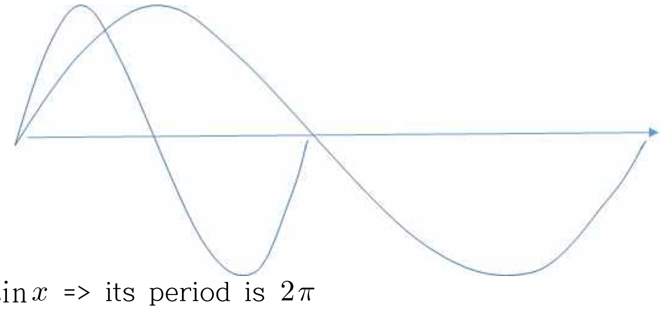
(a) Stretching in y direction



$y = \sin x \quad \Rightarrow y = 2\sin x$

$y = \sin x \quad \Rightarrow y = \frac{1}{2} sinx$

(b) Stretching in x direction

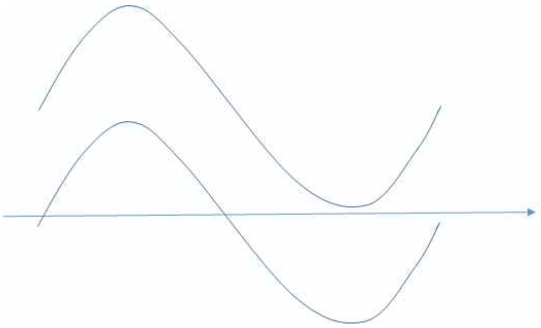


$y = \sin x$ => its period is 2π
$y = \sin 2x$ => its period is $2\pi/2 = \pi$

$y = \sin\frac{1}{2}x$ => its period is $2\times 2\pi = 4\pi$

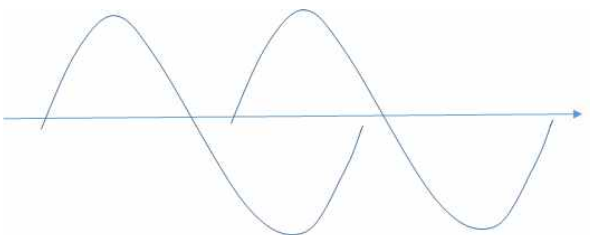
$y = \sin kx$ => period is $\frac{2\pi}{k}$

(c) Translation in y direction



$y = \sin x \Rightarrow y = 1 + \sin x$

(d) Translation in x direction



$y = \sin x \Rightarrow y = \sin(x - \pi)$ (shift to the right by π)

Example) sine function has amplitude 3, largest value =6, smallest value =0, period= π =>?

$y = 3\sin x \Rightarrow y = 3\sin 2x$ (between 3 and -3)
amplitude chosen => vertical stretching
period of sin kx => 2pi/k. Put 2pi/k =pi =>k=2
multiply by 2 => horizontal streching => period = $2\pi/2 = \pi$
It oscillates between 6 and 0. => average line is y=3 => vertical shifting by 3
=> $y = 3 + 3\sin 2x$
Exercise) Sketch its graph.

(2) Example

Plot the annual change of daylight hours of Seoul. Annual data of monthly average daylight hours are followings:

Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
09:52	10:44	11:53	13:08	14:10	14:42	14:28	13:34	12:22	11:08	10:06	09:36

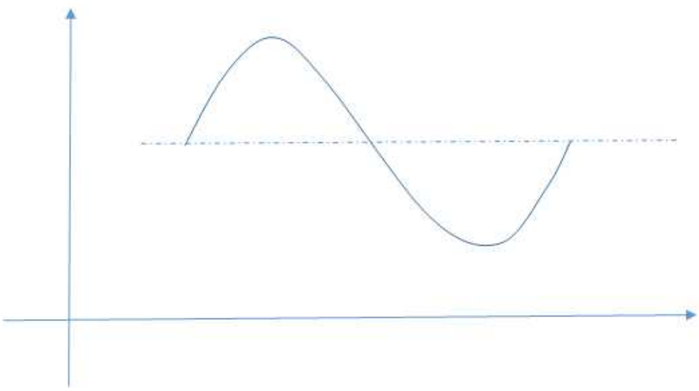
Fit the data using $y = a + b\sin k(t - p)$ where t is measured in days. Decide a, b, k, p.

=> The period is 365 days, thus $k = 2\pi/365$=0.0172.

=> Annual average of daylight hours is 12, thus set $a = 12$.

=> To decide the amplitude, find longest and shortest. Then $b = 1/2(14\frac{42}{60} - 9\frac{36}{60})$.

=> The daylight hours is about 12 on March 20, thus take $p = 88$.



(horizontal axis: time (days), vertical axis: daylight hours)

2. Derivatives of sine and cosine functions

Motivation) In the example of daylight hours of Seoul, can we find when the daylight length most rapidly decreases or increase?

By definition, derivative of sine function is written as

$$\frac{d}{dx}sinx = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

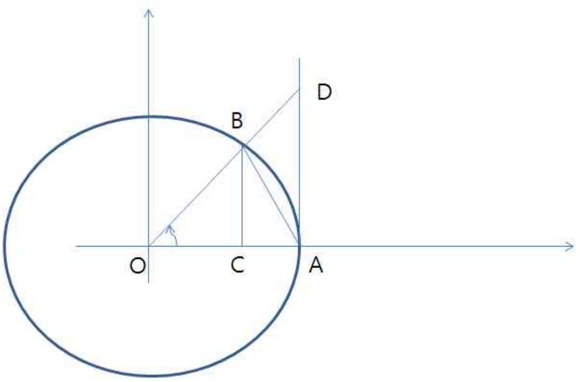
$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\sin x\,\frac{\cos h - 1}{h} + \cos x\,\frac{\sin h}{h}\right)$$

$$= \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

The problem is reduced to evaluate $\lim_{h\to 0}\frac{\cos h - 1}{h}$ and $\lim_{h\to 0}\frac{\sin h}{h}$.

(Theorem)** $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

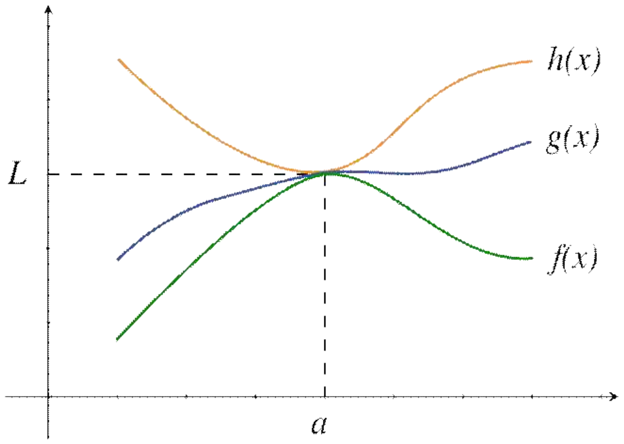


Proof. We will show that $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$ for $-\pi/2 < \theta < \pi/2,\ \theta \ne 0$.

If this is true, then by sandwich theorem (or squeeze theorem) we get the desired consequence.

RECALL **sandwich theorem (or squeeze theorem)**

If $f(x) \le g(x) \le h(x)$ over $(a-\epsilon, a) \cup (a, a+\epsilon)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$



We first prove that $\sin\theta \le \theta$ for $0 < \theta < \pi/2$.
In picture above, we take two points A and B on the unit circle. The angle between A and B is θ. Draw the perpendicular from B to line OA. Denote its foot with C. Consider the triangle BOC. Then we have

$$\sin\theta = BC/OB = BC \le AB \le \widehat{AB} = \theta.$$

Next, we will prove that $\theta \le \tan\theta$ for $0 < \theta < \pi/2$.
Consider a tangent line to the circle whose part is arc AB at the point A. Extend the line OB and denote the intersection with the tangent line at A with D. Compare the area of the sector OAB with the area of the triangle OAD.

=> $\frac{1}{2}\overline{OA}^2\theta \le \frac{1}{2}\overline{OA}\,\overline{AD}$

$$\Rightarrow \frac{1}{2}(1)^2\theta \le \frac{1}{2}(1)\tan\theta$$

$$\Rightarrow \theta \le \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \cos\theta \le \frac{\sin\theta}{\theta}$$

(Squeeze theorem =>) limit of lower bound function (cosine theta) =1 as theta ->0 and limit of upper bound function (=1) => both limits coincide => $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Theorem $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

(Remark : it is the slope of the tangent lint to cosine graph at (0, 1))

Hint: try to multiply $\cos h + 1$ to the top and the bottom, then simplify the expression.

$$\lim_{h \to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$

where $\cos^2 h + \sin^2 h = 1$ is used.

Exercise) Completes the proof. (We can use $\lim_{h \to 0} \frac{\sin h}{h} = 1$)

Theorem $\frac{d}{dx} \sin x = \cos x.$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right)$$

$$= \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right)$$

$$= \sin x(0) + \cos x(1) = \cos x$$

Exercise)

(a) Find the derivative of $y = \cos x$ applying definition of derivative to cosine function.

(b) Find the derivative of $y = \tan x$ (using quotient rule)

Exercise) In example of daylight hours of Seoul, when is the length of daylight hours most rapidly increasing (rep. decreasing)? How rapidly is the daylight hours increasing on that day?

=> need to find derivative of $y = a + b\sin k(t-p)$

=> Need chain rule to answer as well as derivative of sine function.