Module Derivative of exponential function

1. Exponential function

Example) The population of Mexico in the early 1980s is given as follows.

Year	Population (millions)	Change in population (millions)
1980	67.38	
1981	69.13	1.75
1982	70.93	1.80
1983	72.77	1.84
1984	74.66	1.89
1985	76.60	1.94
1986	78.59	1.99

Question: Find population function $P = P(t)$.

Is it linear function?

$P(t) = at + b$

=> P(t+1)-P(t) = a : difference is constant

Ratio of each year's population to the previous year's population.

(Population in 1981)/(Population in 1980)= 69.13/67.38 =1.026

(Population in 1982)/(Population in 1981)= 70.93/69.13 =1.026

Population t years after 1980 is given by $P = 67.38(1.026)^t$.

(1) Definition

For a given positive constant a which is not 1, the function $f(x) = a^x$ is defined on the set of all real numbers. It is called exponential function and the number a is called its base.

Example. Consider $f(x) = 2^x$. Its values as follows

x	2^x
-2	$2^{-2} = 1/2^2 = 1/4$
-1	$2^{-1} = 1/2$
0	$2^0 = 1$

1	2
2	$2^2=4$

For a rational number $x=p/q$, $2^{p/q}=\sqrt[q]{2^p}$.

For an irrational number x, take a sequence of rationals (x_k) such that $\lim_{k\to\infty} x_k = x$. Then define $2^x = \lim_{k\to\infty} 2^{x_k}$. The exponential function is continuous function.

Exercise. Determine the domain and range of $f(x)=(0.3)^x$. Is it increasing or decreasing?

(2) Exponential rules $(a,b>0,\ a,b\neq 1,\ x,y\in R)$)

1. $a^{x+y}=a^x a^y$
2. $a^{x-y}=\dfrac{a^x}{a^y}$
3. $(a^x)^y=a^{xy}$
4. $(ab)^x=a^x b^x$

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2. Derivative of $y=a^x$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=a^x\left[\lim_{h\to 0}\frac{a^h-1}{h}\right]$$

$$=a^x\left(\frac{d}{dx}a^x|_{x=0}\right)$$

Derivative of a^x is Ca^x where C is a constant to be evaluated.

The constant C is turned out to be derivative of a^x at x=0.

If a=2, then $\lim_{h\to 0}\frac{2^h-1}{h}\approx 0.69$ and if a=3, then $\lim_{h\to 0}\frac{3^h-1}{h}\approx 1.10$.

(*) Since $\lim_{h\to 0}\frac{a^h-1}{h}$ increases if $a>1$ and a increases, there should be a number a for which $\lim_{h\to 0}\frac{a^h-1}{h}=1$ and that number is between 2 and 3.

Denote that number with e.

(1) Definition) The number e satisfies $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, thus $\frac{d}{dx} e^x = e^x$. It is called natural constant (e after initial of Euler).

(2) Question: Can we evaluate the number e more accurately?

From $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, we have $e^h \approx 1 + h$ for very small number h and $e \approx (1+h)^{1/h}$. Thus we can define $e = \lim_{h \to 0} (1+h)^{1/h}$ or equivalently $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.

(Alternative definition for natural constant)

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

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(3) sequence

(1+1)=2

$(1+1/2)^2 = 1 + 1 + \frac{1}{4} = 2.25$

$(1+1/3)^3 = 1 + 1 + 1/3 + 1/27 = 2 + \frac{10}{27}$

This sequence is increasing and bounded above by 3

$e \approx 2.718$

It is known that it is irrational number (further more transcendental number like π).

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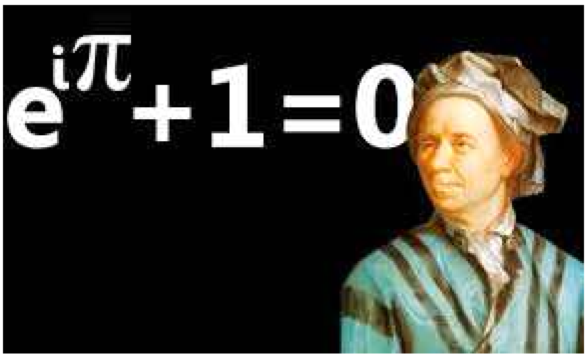
3. Example at which natural constant naturally comes up

(ref Stewart section 6.5)

Jacob Bernoulli

(Bernoulli family is famous for several outstanding mathematicians)

Euler was pupil of his younger brother Johan Bernoulli.



Question: If P Korean won is deposited in an account at an annual interest rate of r, what is balance after t years?

The answer depends on the number of times the interest is compounded.

Suppose that initial deposit in your account is $P = 1000$ won and annual interest rate is 8%. If it is compounded annually, the balance after 1 year is

$$1000 + 1000 \times 0.08 = 1000(1 + 0.08) = 1080 \text{ won.}$$

What if it is compounded semi-annually? It means that the interest is compounded twice in a year, 4% each. Thus the balance after 1 year is

$$\begin{aligned}&1000(1+0.08/2)+1000(1+0.08/2)(0.08/2)\\&=1000(1+0.08/2)^2 = 1081.6\end{aligned}$$

What if it is compounded quarterly? It means that the interest is compounded four times in a year, 2% each. Thus the balance after 1 year is

day	balance
3/31	$1000(1+0.08/4)$
6/30	$1000(1+0.08/4)+1000(1+0.08/4)(0.08/4)=1000(1+0.08/4)^2$
9/31	$1000(1+0.08/4)^2+1000(1+0.08/4)^2(0.08/4)=1000(1+0.08/4)^3$
12/31	$1000(1+0.08/4)^3+1000(1+0.08/4)^3(0.08/4)=1000(1+0.08/4)^4$

In general, the interest is compounded M times yearly, then the value of account after 1 year is $1000\left(1+\frac{0.08}{M}\right)^M$.

P won is deposited in an account at an annual interest rate of r. If it is compounded M times yearly, then the value of account after t years is $1000\left(1+\frac{0.08}{M}\right)^{tM}$.

In general, interest rate is r.
What if it is compounded every micro second? Then we have

$$\lim_{M\to\infty} P(1+r/M)^{tM}$$

$$= P\lim_{x\to 0}[(1+x)^{\frac{1}{x}}]^{tr} = Pe^{rt}$$

where $x = r/M$ and as $M \to \infty$, $x \to 0$.

It is called the balance after t years of continuous compounding.

Example) Initial deposit is 1000 dollars. Interest rate is 6%. What is the balance after 3 years under continuous compounding?

$P(t) = P_0 e^{rt}$

$P(3) = 1000\, e^{0.06 \times 3} = 1000\, e^{0.18} \approx 1197.22$
