Module Definite integral: Fundamental theorem of Calculus

- Fundamental theorem of Calculus
- Evaluation of definite integral
- Application of definite integral

Definite integral $\int_a^b f(x)dx$ =limit of Riemann sum for function f and interval [a, b]

=> $\sum_{i=1}^{n} f(x^*_i)\Delta x_i$

1. Issue of evaluation

For $f:[a,b]\to \boldsymbol{R}$, Area function is defined as follows: $A(x)=\int_a^x f(t)dt$ on $a \le x \le b$.

Fundamental theorem of Calculus A. If f is continuous on [a, b] then area function A is differentiable on (a, b) and $A'(x)=f(x)$

Fundamental theorem of Calculus B. (Corollary of FTC A) If f is continuous on [a, b] then $\int_a^b f(x)dx = F(b)-F(a)$ where F is a function such that $F'=f$.

(IDEA) $\int_a^b f(x)dx = A(b)$

$A'(x)=f(x)=F'(x)$ => $\frac{d}{dx}(A(x)-F(x))=0$ over a< x <b

=> $A(x)-F(x)=const$

(Following is used: If derivative of a certain function is zero over an interval, then this function is constant)

$$\int_a^b f(x)dx = A(b)-A(a) = (F(b)-const)-(F(a)-const) = F(b)-F(a)$$

Definition) Anti-derivative of f ⇔ A function F is called an anti-derivative of f if derivative of F is f.

It is denoted with $\int f(x)\,dx$ (also called indefinite integral of f)

*Anti-derivative of f is unique up to constant.

Example) Evaluate $\int_0^{\pi/2} \cos x\,dx$. Antiderivative of $\cos x$ is $\sin x$, thus by FTC we have

$$\int_0^{\pi/2} \cos x\,dx = \sin x|_0^{\pi/2} = \sin\pi/2 - \sin 0 = 1$$

Example) What is wrong with $\int_{-1}^{1} \frac{1}{x^2}dx = -\frac{1}{x}|_{-1}^{1} = -2$?

2. Rules for Anti-derivative/Indefinite integrals (section 4.4)

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
2. $\int \cos x dx = \sin x + C$
3. $\int \sin x dx = -\cos x + C$
4. $\int \alpha f(x) + \beta g(x) dx = \alpha \int f(x) dx + \beta \int g(x) dx$

Remark. $\int x^{-1} dx = \int \frac{1}{x} dx$ is in fact a certain logarithmic function. We will consider this problem later.

Example) (Finding anti-derivative)

$$\int 7x^2 + 4x + 5\ dx = 7\int x^2\ dx + 4\int x dx + 5\int dx$$

$$= \frac{7}{3}x^3 + \frac{4}{2}x^2 + 5x + C$$

Example) Evaluate $\int_1^2 \sqrt{x} - \frac{1}{\sqrt{x}} dx$

Here indefinite integral $\int \sqrt{x} - \frac{1}{\sqrt{x}}\, dx = \int x^{1/2} dx - \int x^{-1/2} dx$

$$= \frac{1}{1/2+1} x^{1/2+1} - \frac{1}{-1/2+1} x^{-1/2+1} + C$$

$$= \frac{2}{3} x^{3/2} - 2x^{1/2} + C$$

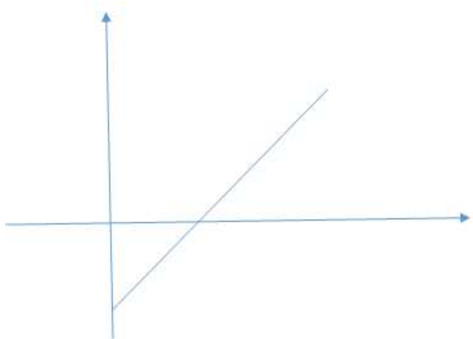
By FTC, $\int_1^2 \sqrt{x} - \frac{1}{\sqrt{x}} dx = (\frac{2}{3} x^{3/2} - 2x^{1/2})|_{x=1}^{x=2}$

$$= (\frac{2}{3} 2\sqrt{2} - 2\sqrt{2}) - (\frac{2}{3} - 2)$$

$$= -\frac{2}{3}\sqrt{2} + \frac{4}{3}$$

3. Application of definite integral

Example) Average temperature of the steel rod with the length 2 meter. Temperature distribution $T = T(x) = 5(x - 0.8)$ over $0 \le x \le 2$.



Average of temperature for n sample point chosen from the rod is

$$\frac{1}{n} \sum_{k=1}^{n} T(x_k) = \frac{1}{2} \sum_{k=1}^{n} T(x_k) \frac{2}{n} \quad (**)$$

Here we choose points $x_k, k = 0, 1, \ldots, n$ as follows: $x_k = k\frac{2}{n}$. Point are equally spaced. It can be achieved by taking the distance between two consecutive points equal to (entire length)/(number of division) = 2/n.

We can interpret average of n points (**) as a 1/2 times righthand sum of temperature distribution function $T(x)$ over $0 \le x \le 2$.

Thus

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}T(x_k)=\lim_{n\to\infty}\frac{1}{2}\sum_{k=1}^{n}T(x_k)\frac{2}{n}=\frac{1}{2}\int_0^2 T(x)dx$$

Conclusion:

$$\text{Average temperature}=\frac{1}{2}\int_0^2 T(x)dx$$

Exercise) Evaluate the definite integral above.

Problem) Linear density of a metal rod of length 4 meter is given by $f(x)=9+2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total weight of the rod.

Solution) weight = density times length (see unit of density is kilogram/meter)
For instance, if the length of the rod is 2 meter and its density is 0.5 kg/m, then total weight is $0.5kg/m\times 2m=1kg$.
In our problem, density is not constant. It is different point by point.
We can find approximate weight as follows. Place the rod on the x-axis, then it is placed over [0, 4]. For instance, divide the interval 10 segments. Then length of the segment is 4/10=0.4. Thus [0, 4] is divided into [0, 0.4], [0.4, 0.8], ...[3.6, 4].

j th interval is [(j-1)0.4, j 0.4]. Take a representative value for linear density. For instance, we take density at the right-end point $f(x_j)=f(0.4j)=9+2\sqrt{0.4j}$. Then the weight of the segment [(j-1)0.4, j 0.4] is $f(x_j)\Delta x=(9+2\sqrt{0.4j})0.4$ where Δx is the length of the segment. Now add the weight of all the segments

$$\sum_{j=1}^{10}f(x_j)\Delta x=\sum_{j=1}^{10}(9+2\sqrt{0.4j})0.4$$

Can you recognize this? It is just right-hand sum R_{10} for f and [0, 4]. In general we divide the rod into n segment of which length is equal, then

$$\sum_{j=1}^{n}f(x_j)\Delta x=\sum_{j=1}^{n}f(0+j\Delta x)\Delta x=\sum_{j=1}^{n}f(j\frac{10}{n})\frac{10}{n}=\sum_{j=1}^{n}(9+2\sqrt{(10j/n)})(10/n)$$

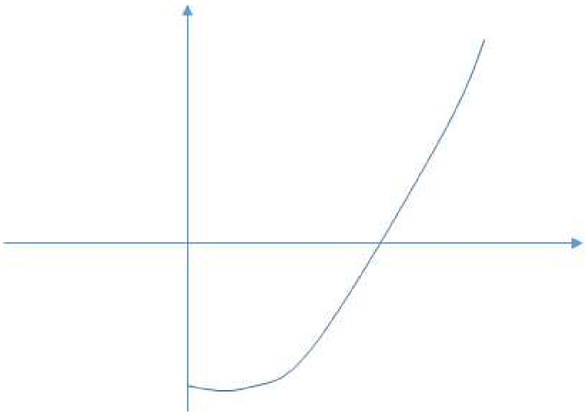
approximates the weight of the rod.
Thus the real weight of the rod is the limit of the above as $n\to\infty$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(x_j)\Delta x=\int_0^4 f(x)dx=\int_0^4 9+2\sqrt{x}\;dx$$

4. Distance problem revisited

Example) A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (meter/sec).

(a) What is the displacement (net change in position) of the particle over $1 \le t \le 5$?



In the beginning, the particle was moving the left. Note that $v(t) = (t-3)(t+2)$ it changes its direction to the right at t =3.

Displacement is defined to be $\int_{t_1}^{t_2} v(t)dt$

$$\int_1^5 t^2 - t - 6\, dt = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t \Big|_{t=1}^{t=5}$$

$$= \frac{1}{3}(125-1) - \frac{1}{2}(25-1) - 6(5-1) = \frac{124}{3} - 36 = 41.3 - 36 = 5.3$$

It means that from the position at t=1 after 4 seconds, the position of the particle is shifted to the right by 5. meter as result of total motion.

(b) What is the distance traveled during $1 \le t \le 5$?

=> Area of the region bounded by t=1, t=5, v-axis and the v=v(t) graph.

$$\int_1^5 |v(t)|\, dt = \int_1^3 -(t^2 - t - 6)\, dt + \int_3^5 t^2 - t - 6\, dt$$

Net change theorem (section 4.4)

For given function $y = F(x)$, the integral of rate of change $\frac{dF}{dx}$ is the net change

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

Example)

V(t)=the volume of water in a reservoir at time t. $\frac{dV}{dt}$= rate at which water flows into the reservoir at time t

=> $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$: net change in amount of water in the reservoir between $t = t_1$ and $t = t_2$