Module Concavity and second derivatives

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Question) What are the extreme values of $f(x) = (x^2 - x)^{2/3}$ over entire real line?

We can compare function values of several local maximum points. But the biggest value of them may not be the absolute max. (abs max may not exist in open interval)
=> We need to know entire behavior of function.

Definition) The function f is **increasing (resp. decreasing)** on $a < x < b$ if $f(x) < f(y)$ (resp. $f(x) > f(y)$) whenever $x < y$.

Example) $f(x) = x^3$ is increasing on the real line. ($x^3 < y^3$ if $x < y$)

Its average rate of change $\frac{f(b) - f(a)}{b - a} > 0$ for any $a < b$.

1. Increasing/Decreasing Test

Suppose that f is defined and differentiable on $a < x < b$. Then f is increasing (resp. decreasing) if $f'(x) > 0$ (resp. $f'(x) < 0$) on $a < x < b$.

Example) $f(x) = \frac{1}{x} + \frac{1}{1-x}$, $0 < x < 1$. Decide where it increase and where it decrease.

Step 1. Find the derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{(x-1)^2} = \frac{2x-1}{x^2(x-1)^2}$

Step 2. Find the critical number => x=1/2

Step 3. Divide the domain using the critical number => $0 < x < 1/2$,$1/2 < x < 1$

Step 4. Make a table (sign check of derivative)

interval	2x-1	$f'(x)$	I/D
0< x < 1/2	neg	neg	Dec
1/2 < x< 1	pos	pos	Inc

Implication => f has abs min at x=1/2

Example)

$f(x)=(x^2-x)^{2/3}$ on entire real line. Decide where it increase and where it decrease.

Step 1 $f'(x)=(2/3)\frac{2x-1}{\sqrt[3]{x^2-x}}=(2/3)\frac{2x-1}{\sqrt[3]{x(x-1)}}$

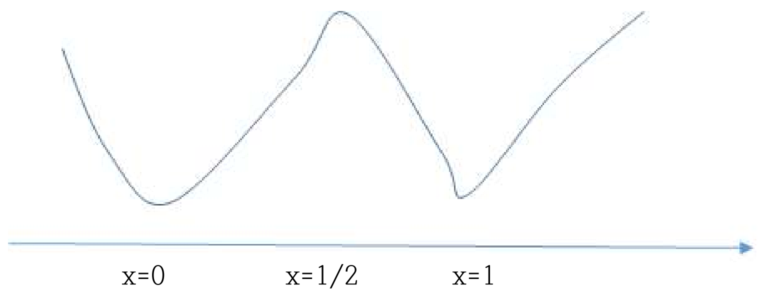
Step 2 Critical numbers: 2x-1=0 => x=1/2, $x^2-x=0$ => x=0 , 1

Step 3 Divide the real line using x=0, 1/2, 1

=> $(-\infty,0)$, $(0,1/2)$, $(1/2,1)$, $(1,+\infty)$

Step 4 Table

Interval	x	2x-1	x-1	f'	I/D
x<0	-	-	-	-	Dec
0< x<1/2	+	-	-	+	Inc
1/2 <x < 1	+	+	-	-	Dec
x >1	+	+	+	+	Inc



Implication => From this observation, we can conclude that f has a absolute minimum at $x=0$ and $x=1$. To decide whether f has absolute max at x=1, we need to check asymptotic behavior of f.

$\lim_{x\to+\infty}(x^2-x)^{2/3}=+\infty$ => f does not have abs max.

x^2 -x = x^2 (1- 1/x) -> x^2 (inc without bound)

2. Second derivative

We can consider derivative of $f'(x)$ => $\frac{d}{dx}f'(x) = f''(x) = f^{(2)}(x)$. (2nd derivative of f)

Example) $f(x) = \sqrt{x}$ $=> f'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2}$

$$=> f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\frac{1}{2}x^{-1/2} = \frac{1}{2}(-1/2)x^{-3/2} = -\frac{1}{4}x^{-3/2}$$

Example) $g(x) = x^{3/2}$ => $g'(0)$ exists, but $g''(0)$ does not exist

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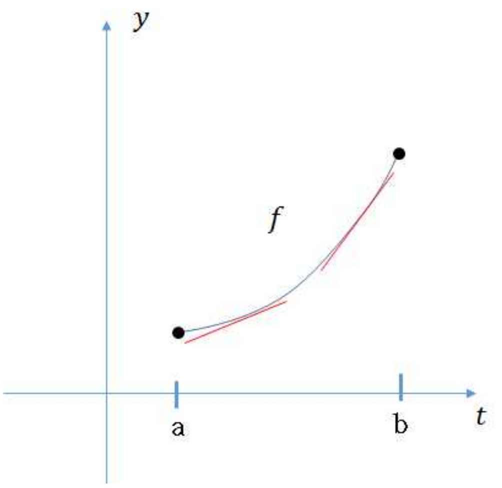
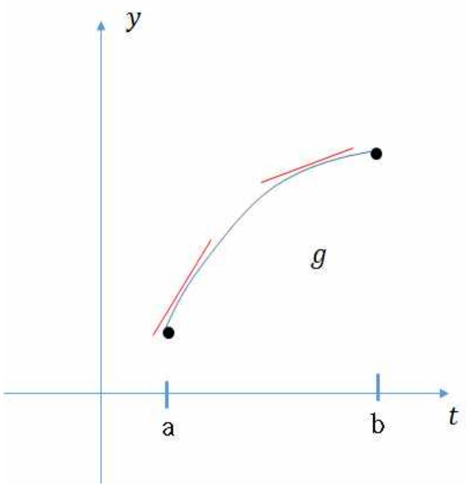
Second derivative of f is rate of change of f'. It measure graph of f becomes steeper or slow down.

f'' gives more information about graph of f

=> Notion of concavity





**Two different increasing patterns

Def) The graph of f lies above all its tangents on an internal (a, b) =>

the graph of f is said to be concave upward on (a, b)

The graph of g lies below all its tangents on an internal (a, b) =>

the graph of g is said to be concave downward on (a, b)

(1) Concave upward case : slope of tangent line is increasing

$\Rightarrow f'$ is increasing

$\Rightarrow (f')' = f'' > 0$

(2) Concave downward : slope of tangent line is decreasing

$\Rightarrow f'$ is decreasing

$\Rightarrow (f')' = f'' < 0$

** Second derivative can be used to check the concavity of the graph of function

3. "Concavity test"

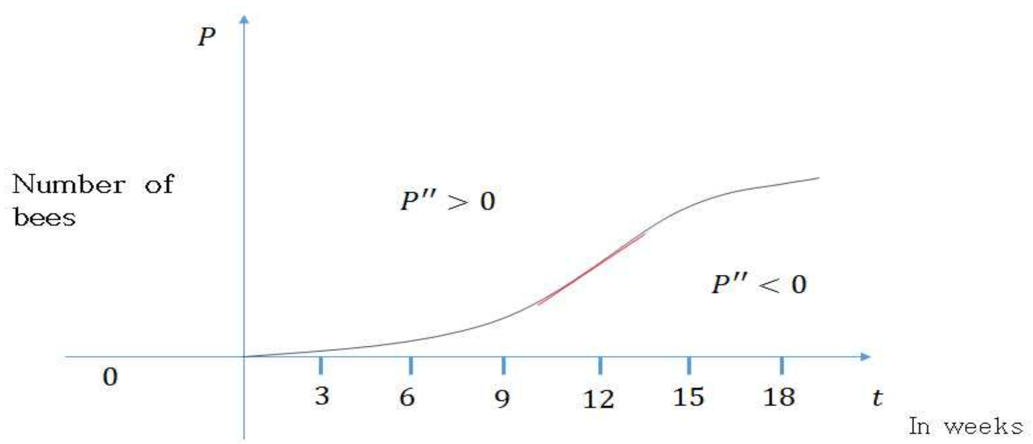
(a) $f'' > 0$ on I $\Rightarrow$ the graph of f is concave upward on I

(b) $f'' < 0$ on I $\Rightarrow$ the graph of f is concave downward on I

Example) $y = \sqrt{x}$ is concave downward on $(0, \infty)$ because $y'' = \frac{d}{dx}\frac{1}{2}x^{-1/2} = -\frac{1}{4}x^{-3/2} < 0$ on $(0, \infty)$.

Example)

A population graph for Cyprian honey bees



The number increases slowly in the beginning. After weeks, it increases faster and faster, that it P increase ($P'' > 0$) until $t = 12$.

After then it begins to slow down (even though it increase) $t = 12$ is a point where sign (P'') changes. The graph is convex before t=12, and concave after t=12.

Definition

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

Example) Discuss concavity of the graph of $y = \sin x$ over $0 < x < 2\pi$. Find the points where its second derivative vanishes. We have $y'' = -\sin x = 0$ at $x = \pi$.

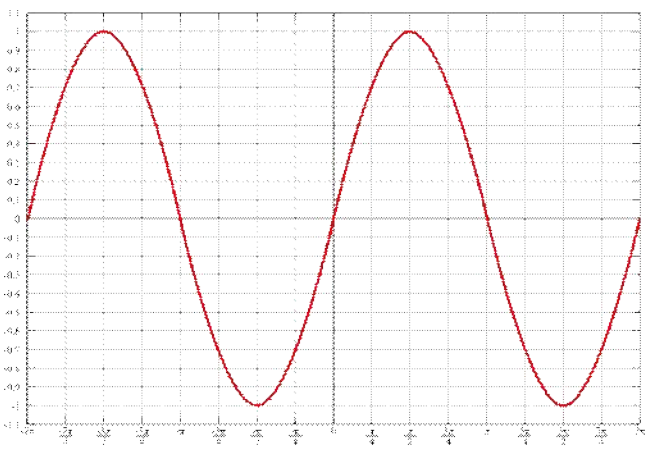
Then look at the sign change of y'':

$$-\sin x < 0, \quad 0 < x < \pi$$

$$-\sin x > 0, \quad \pi < x < 2\pi$$

Thus the graph is concave downward on $0 < x < \pi$ and concave upward on $\pi < x < 2\pi$.

Another aspect of the fact that $y'' = -\sin x = 0$ at $x = \pi$ is that $x = \pi$ is a critical number of $y' = \cos x$. The derivative of y has minimum there. It means that $y = \sin x$ decreases most rapidly at $x = \pi$.



Remark) Second derivative of certain function may not exist at its inflection point even if the sign of second derivative changes around that point.

Example) $y = \sqrt[3]{x}$ => $y' = \frac{1}{3}x^{-2/3}$, $y'' = -\frac{2}{9}x^{-5/3} = -\frac{2}{9}\frac{1}{\sqrt[3]{x^5}}$

=> second derivative does not exist at x=0

sign of y'' changes around x=0 => (0,0) is inflection point

Example (Concavity test and shape of graph)

$y = \frac{x^3}{x^2+1}$ => $y' = \frac{x^2(x^2+3)}{(x^2+1)^2}$ => $y'' = \frac{2x(3-x^2)}{(x^2+1)^3}$

Step 1. Critical numbers of y' (candidate for inflection points)

$y'' = 0$ => $2x(3-x^2) = 0$ => $x = 0, \sqrt{3}, -\sqrt{3}$

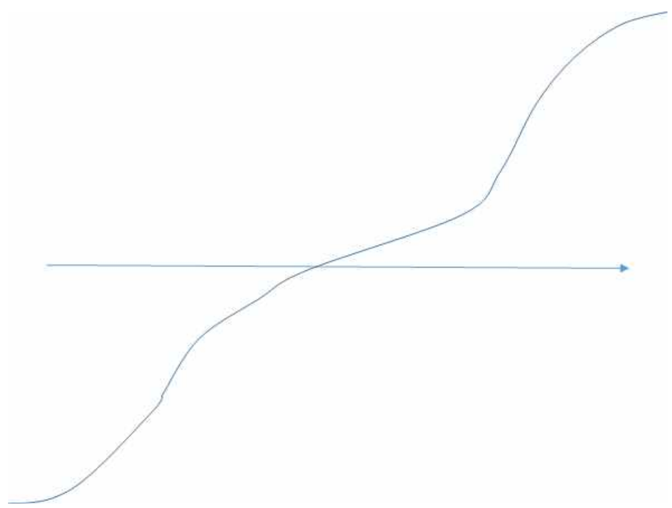
Second derivative is defined everywhere.

Step 2. Divide the domain with these points

$(-\infty, -\sqrt{3}),\ (-\sqrt{3}, 0), (0, \sqrt{3}), (\sqrt{3}, +\infty)$

Step 3. Table for concavity test

Interval	x	$3-x^2$	y''	concavity
$x < -\sqrt{3}$	-	-	+	CU
$-\sqrt{3} < x < 0$	-	+	-	CD
$0 < x < \sqrt{3}$	+	+	+	CU
$x > \sqrt{3}$	+	-	-	CD



Inflection points => $(-\sqrt{3}, -3\frac{\sqrt{3}}{4})$, $(\sqrt{3}, 3\frac{\sqrt{3}}{4})$, (0,0)

Exercise) Decide where the curve $y = \frac{x^2}{x-1}$ is concave upward and where it is concave downward. Does it have inflection points?