Module Chain Rule

1. Chain Rule

Motivation: How can we find the derivative of $y=\sqrt{x^2+1}$?

=> It is a composition of $y=\sqrt{z}$ with $z=x^2+1$. What can we say about $\frac{dy}{dx}$ in terms of $\frac{dy}{dz}$ and $\frac{dz}{dx}$?

(1) Motivating Example

We release a ballon with light gas on air. It keep going upward. As it goes up, its volume changes because air pressure is different on each altitude.
Question: At which altitude, does the ballon blow out?

How rapidly does the volume of the ballon changes at each altitude?
=> a function $V=V(h)$ representing the volume V (in liter) of the balloon at each altitude h (in kilometer). We want to find $\frac{dV}{dh}$.

=>(1) The volume of the gas is inversely proportional to pressure
(2) The air pressure (P) is linearly decreasing function of the height.

=> $V=C/P$ and $P=ah+b$.

We don't have to know the functions explicitly to get dV/dh at given altitude.

Find the dV/dh at a specific height, say h_0.
Suppose that
-the air pressure at h_0 is P_0.
-the rate of the decrease of the pressure with respect to the height is -20 Pa/km
-the rate of the decrease of the volume of gas with respect to the pressure at P_0 is known to be -0.1 liter/Pa (1Pa=1 Newton/m^2).

=> $\Delta P \approx -20\Delta h$ (Pressure decreases 20 times as much as height increases)
$\Delta V \approx = -0.1\Delta P$ (Volume decreases 0.1 times as much as Pressure increases)

Question: $\frac{\Delta V}{\Delta H}=?$

$\Delta V \approx -0.1\Delta P \approx (-0.1)(-20)\Delta h = 2\Delta h$

=> $\frac{\Delta V}{\Delta h} \approx 2=$ (-0.1) (-20) $\approx \frac{\Delta V}{\Delta P} \times \frac{\Delta P}{\Delta h}$

=> We expect:

$$dV/dh|_{h=h_0} = dV/dP \times dP/dh = (-0.1\ L/Pa) \times (-20\ Pa/km) = 2\ L/km$$

We can say that the ballon expands by about 2 liter as it moves up by 1 km at the height h_0.

(Chain Rule) The differentiable functions $y=f(u)$ and $u=g(x)$ are given. Then the derivative of $y=f(g(x))$ is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(g(x))g'(x).$$

Example) $y=(x^2+1)^{100}$. Set $y=u^{100}$, $u=x^2+1$. (Identify f and g in composition)

=> u: intermediate variable

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 100u^{99}(2x) = 100(x^2+1)^{99}(2x) = 200x(x^2+1)^{99}.$$

(Simple application of chain rule)

Identify <u>outer function</u> and <u>inner function</u>. Inner function is usually given inside the parenthesis or radicals.

The chain rule :

the derivative of the whole function

=(the derivative of outer function evaluated at inner function) $\times$ (the derivative of inner function)

Back to previous example $y=(x^2+1)^{100}$. To find dy/dx

Outer function $(\ \ )^{100}$, and inner function x^2+1

=> $\frac{dy}{dx} = 100(x^2+1)^{99}(x^2+1)' = 100(x^2+1)^{99}(2x).$

Example) Find the derivative of $y = \frac{1}{\sqrt{2-x^3}}$.

Here its outer function is $\frac{1}{\sqrt{(\ \ )}}$ (equivalently $(\ \ )^{-1/2}$) and inner function is $2-x^3$. The derivative of outer function is $-\frac{1}{2}(\ \ )^{-3/2}$ by the power rule and the derivative of inner function is $-3x^2$. Thus

$$\frac{dy}{dx} = -\frac{1}{2}(2-x^2)^{-3/2}(-3x^2) = \frac{3}{2}x^2(2-x^2)^{-3/2}.$$

(Alternatively)

$y = \frac{1}{\sqrt{u}}, \ u = 2-x^3$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \quad y = u^{-1/2}, \ \frac{dy}{du} = -\frac{1}{2}u^{-3/2}, \ \frac{du}{dx} = -3x^2$

=> $\frac{dy}{dx} = (-1/2)u^{-3/2}(-3x^2) = \frac{3}{2}x^2(2-x^3)^{-3/2}$ = $\frac{3}{2}\frac{x^2}{(\sqrt{2-x^3})^3}$

Exercise

1. Find $\frac{dy}{dx}$ for following functions using chain rule.

(a) $y = \sqrt{x^3-1}$

(b) $y = \frac{1}{(x^4-3x)^7}$

(c) $y = \left(\frac{x^2-1}{x^4+1}\right)^{20}$

2. For $f(x) = \frac{1}{x^3-5x^2+7}$, find $f'(x)$ using the quotient rule and then using the chain rule. Compare two method.
